Is the HECM Program Sustainable?

Evidence from Pricing Mortgage Insurance Premiums and Non-Recourse Provisions Using Conditional Esscher Transform

Hua Chen, Samuel H. Cox, and Shaun S. Wang

Hua Chen (Corresponding author)
Department of Risk, Insurance, and Health Management
Temple University
487 Ritter Annex
Philadelphia, PA 19122
Phone: (215) 204-5905
Fax: (215) 204-4712
Email: hchen@temple.edu

Samuel H. Cox
Warren Centre for Actuarial Studies and Research
University of Manitoba
181 Freedman Crescent
Winnipeg, Manitoba R3T 5V4, Canada
Phone: (204) 474-7426
Fax: (204) 474-7545
Email: scox@cc.umanitoba.ca

Shaun S. Wang
Department of Risk Management and Insurance
Georgia State University
P.O. Box 4036
Atlanta, GA 30302-4036
Phone: (404) 413-7486
Fax: (404) 413-7499
Email: inssww@langate.gsu.edu

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Abstract:

The ongoing subprime mortgage crisis depreciates house values and therefore has major adverse impacts on reverse mortgages. The purpose of this paper is to build a modeling and pricing framework to investigate the sustainability of HECM programs in the U.S. Traditional HECM models use period life tables, neglecting the dynamics of mortality rates and extreme longevity events. Also, previous research assumes house prices are driven by a geometric Brownian motion, contradicting with the fact that housing price exhibits strong autocorrelation and varying volatility over time. We propose a generalized Lee-Carter model with permanent jump effects to fit the actually mortality data, and model the house price index via an ARIMA-GARCH process. We employ the conditional Esscher transform to price the non-recourse provision of reverse mortgages and compare it with calculated mortgage insurance premiums. Our results indicate that the HECM program is sustainable.

1. Introduction

According to the statistics from the U.S. Census Bureau, more than 34 million Americans live above age 65. That number is expected to increase to 71 million by the year 2030, which will account for 19.6% of the population.[1] With the retirement of baby-boomers and the increase in the share of the elderly in the population, the retirement programs in the U.S. face an "aging-population tsunami" and significant future imbalances consequently. In addition, the shift of pension plans from defined benefit (DB) to defined contribution (DC) and the declining contribution levels from employers impose big challenges on financial budgets of the aged population after their retirement. The elderly may not only receive reduced monthly incomes, but also experience rising health-care costs and decreasing pension plan benefits. It is increasingly difficult for them to maintain financial independence and the standard of living. However, the American Housing Survey shows that more than 12.5 million elderly have no mortgage debt, and the median value of the unmortgaged homes is $127,959.[2] "House rich and cash poor" is the phrase often used to describe their dilemma.

Since the 1970s, academics and practitioners have sought to create mortgage instruments to enable elderly homeowners to borrow by using the equity in their homes as collaterals, which are referred to as reverse mortgages. Providers of reverse mortgages advance a lump sum or periodic payments to elderly homeowners. The loans accrue with interests and are settled only when borrowers die, sell or vacate their homes to live elsewhere. Borrowers enjoy the favorable merits of reverse mortgages in several aspects. First, it allows homeowners to convert their home equity into cash flows without having to move out of the properties. Second, borrowers have no

[1] U.S. Census Bureau. International database. Table 094.
Available at http://www.census.gov/population/www/projections/natdet-D1A.html

[2] American Housing Survey for the United States: 2005, Current Housing Reports. H150/05. U.S. Department of Housing and Urban Development and U.S. Census Bureau. Aug. 2006, P156

obligation to repay the loans as long as they are alive or reside in their homes. Finally, the repayments are capped with the proceeds from the sale of the properties. When a loan is terminated, if the loan balance is larger than the property value (usually referred to as crossover risk), the provider recovers only up to the sale price of the property. This is the so-called non-recourse provision.

In 1989, after the Department of Housing and Urban Development (HUD) introduced the Home Equity Conversion Mortgage (HECM) program, reverse mortgages became widely available in the United States. There are three major reverse mortgage programs in the U.S. market, which are the HUD's HECM, Fannie Mae's Home Keeper program, and Financial Freedom's Cash Account Plan. In late 2006, the Lifestyle Plan was introduced in the states of California, Oregon and Washington, and the plan was expected to be available nationally during 2007. Among them, the HECM program is considered the safest and the most popular program in the U.S., since it is insured by the U.S. federal government and accounts for 95 percent of the market share. For this reason, we focus on analyzing the HECM program in this study.

The main purpose of this study is to develop a framework to model the embedded risks, value the non-recourse provision, and calculate the present value of insurance premiums in the HECM program. As pointed out by Phillips and Gwin (1992), the lending feature of reverse mortgages subjects loan providers to multiple risks. An increase of the lifespan of the loan resulted from mortality improvement or reduced mobility rates will impose a higher crossover risk. A rise in interest rates will speed up the rate at which the loan accumulates, and will possibly hit the crossover point earlier. Finally, a depressed real estate market will worsen the value of the home. Most of the existing literature on risk modeling in the HECM program has two major flaws. First, some of them use period life tables, and thus neglect the dynamics of mortality rates (see Tse, 1995; Zhai, 2000; Weinrobe, 1988; Szymanoski, 1994). Moreover, almost all of previous studies ignore longevity risk and do not model the mortality improvement jumps explicitly. We employ a generalized Lee-Carter model with permanent-effect jumps to model the mortality rates and construct dynamic life tables for pricing the non-recourse provision and calculating insurance premiums. Second, traditional HECM models assume house prices are driven by a geometric Brownian motion (e.g., Szymanoski, 1994; Ma, Kim and Lew, 2007; Wang, Valdez and Piggott, 2007). However, we find that house price returns exhibit strong autocorrelation and varying volatility over time, which is inconsistent with the "memoryless" feature of geometric Brownian motions. Therefore, we follow Li, Boyle, Hardy and Tan (2007)'s work by using an ARIMA-GARCH model to fit house price returns.

The non-recourse provision in reverse mortgages can be viewed as offering borrowers a series of European exchange options with different times to maturity (Chinloy and Megbolugbe, 1994). Valuation of the non-recourse provision requires us to create an equivalent martingale measure in an incomplete market, since multiple risks are involved in reverse mortgages. The adoption of the GARCH model for house price returns suggests that the change of probability measure via the conditional Esscher transform be applicable in this circumstance. Till now, there is a substantial literature for option valuations in the GARCH framework since the pioneering work of Duan (1995). He introduces the notion of locally risk-neutral valuation relationship (LRNVR), which preserves the one-period-ahead (local) conditional variances of the stock-price dynamics. Siu, Tong and Yang (2004) adopt the conditional Esscher transform to the valuation of derivatives under the general class of GARCH models. They recover Duan's pricing results, assuming the GARCH innovations are conditionally normal. Li, Boyle, Hardy and Tan (2007) apply Siu et al.'s pricing framework, when the conditional mean equation is given by an ARMA process, to price the No-Negative-Equity-Guarantee in equity release markets in the U.K. Their approach can be extended slightly here to value the non-recourse provision of reverse mortgages when the mean equation follows an ARIMA process.

In order to protect lenders of reverse mortgages from possible losses, the Federal Housing Administration (FHA), which is one part of HUD's Office of Housing, charges insurance premiums from borrowers, and pays insurance claims to lenders in case the loan balance exceeds the equity value at the time of settlement. Theoretically, the present value of expected premiums should be equal to the value of the non-recourse provision, under the actuarial equivalence principle. We examine the premium structure of HECM loans and investigate whether insurance premiums are adequate to cover expected claims. We find that the collected premiums exceed the actuarial present value of claim payments to lenders. The HECM program is sustainable.

The rest of this article is organized as follows. In section 2, we review the basic facts of reverse mortgages, using HECM as an example, and develop the pricing formula for the non-recourse provision and insurance premiums. In section 3, we discuss various risks involved in the HECM program. We build a model for longevity risk and provide details on how to construct a dynamic life table in section 4. We review the literature on house price dynamics and model house price returns as an ARIMA-GARCH process in section 5. In section 6, we discuss the conditional Esscher transform and provide the pricing framework. We value the non-recourse provision and calculate the present value of insurance premiums consequently. Finally, section 7 concludes.

2. The HECM Program and the Non-Recourse Provision

The HECM program was authorized by HUD in the Housing and Community Development

Act of 1987. Since 1990, there are more than 308,000 elderly homeowners taking advantage of the HECM program.[3] The market even booms at an incredible rate recently. According to the national Reverse Mortgages Lenders Association, 37,829 HECM loans were originated in 2004, representing a 109% jump over the previous year and nearly 500% growth since 2001. The growth continued in 2005 when 43,131 HECM loans were approved (Wang, Valdez and Piggott, 2007). Currently, it has become the most popular program of its kind in the U.S. Over 95 percent of all reverse mortgage borrowers choose the HECM products (Ma and Deng, 2006). This section reviews the basic facts of the HECM program, to prepare for further discussions in modeling and pricing sections later on.

2.1. Borrower Requirements

Under the HECM program, borrowers must be at least 62 years of age, living in a single-family property that meets HUD's minimum property standard. They must own their homes free and clear, which implies that any home purchase mortgage must be fully repaid either prior to HECM or from the initial proceeds of the HECM product. Before closing, borrowers must attend counseling where third parties will explain the financial implications of entering into the HECM program as well as other options that may be available.

2.2. Payment Options

These are the payment options that borrowers can choose from:

- Lump sum – The borrower receives the entire available principal limit at closing.
- Line of credit – Installments are made at times and in amounts of the borrower's choosing until the line of credit is exhausted.
- Tenure – Equal monthly payments are made as long as the borrower lives and continues

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[3] Sioris, N. Over 300,000 HECM Reverse Mortgages Closed. June 28, 2007. Available at http://www.letyourhomepayyou.com/2007/06/over-300000-hecm-reverse-mortgages.html

to occupy the property as a principal residence.

- Term – Equal monthly payments are made for a fixed period of months selected by the borrower.
- Modified tenure – Combination of line of credit with monthly payments for as long as the borrower remains in the home.
- Modified term – Combination of line of credit with monthly payments for a fixed period of months selected by the borrower.

2.3. Initial Principal Limit

The initial principal limit (IPL) is the initial loan amount that may be extended to a borrower by a lender, which equals the present value of the monthly payments offered to borrowers (Deutsche Bank, 2007). In the case of a lump sum option, it is the amount of cash advance that a borrower can get at the time of loan origination. It is determined by the age of the borrower, the expected interest rate, and the adjusted property value, which is the minimum of the appraised value of the property and the mortgage limit imposed by FHA for one-family house in that specific area.

2.4. Interest Rate Options

The interest rate charged on the loan may be fixed or adjustable, with annual or monthly adjustments linked to the one-year Treasury bill rate. However, 99% of the HECM loans issued to date had an adjustable interest rate, largely because Fannie Mae (Federal National Mortgage Association), which has purchased nearly all of the loans issued under the program, does not purchase fixed-rate loans.

2.5. The Termination Time

To prevent involuntary displacement of elderly homeowners, the HECM loan does not require any repayment until a borrower sells the property, moves out, or dies. A borrower also has the option of prepaying the loan without any penalty. A foreclosure can only take place when a borrower discontinues paying monthly property taxes and insurance, or fails to maintain the property up to a minimum maintenance level.

2.6. The Non-Recourse Provision

The HECM loan is a non-recourse debt. When the loan terminates, if the net proceed from the sale of the property is sufficient to pay the outstanding loan balance, the remaining cash usually belongs to the borrower or his/her beneficiaries. If the sale proceed is not enough to cover the loan balance, the non-recourse provision prevents the lender from pursuing other assets belonging to the borrower, apart from the house.

Denote L_t the outstanding balance of the loan and H_t the value of the mortgaged property at a random time t. If the loan is due at time t, the borrower pays L_t if $H_t \geq L_t$, and H_t if $H_t < L_t$, under the non-recourse provision. The cash flow function of the borrower can be written as follows:

$$repayment = \begin{cases} -H_t, & H_t < L_t \\ -L_t, & H_t \geq L_t \end{cases}$$

$$= -L_t + \max(L_t - H_t, 0). \qquad (1)$$

Equation (1) means the non-recourse provision is equivalent to writing the borrower an European exchange option which changes the mortgaged property value H_t for the loan outstanding balance L_t. The borrower, therefore, is holding a debt position and an exchange option. At the time of termination, the borrower needs to pay the loan balance, but he or she gets the payoff from the exchange option at the same time. Note the termination time t is random here. More precisely, the non-recourse provision is equivalent to writing the borrower a series of

European exchange options with different times to maturity (Chinloy and Megbolugbe, 1994; Li, Boyle, Hardy and Tan, 2007). Let ω be the highest attained age, the payoff from the non-recourse provision written on a cohort group aged x can be expressed as follows:

$$\text{Value} = \sum_{t=0}^{\omega-x-1} E_Q\left[{}_t\, p_x q_{x+t} e^{-rt} \max(L_t - H_t, 0)\right], \tag{2}$$

where r is the risk-free interest rate, ${}_t\, p_x$ is the probability that an individual aged x will survive another t years, q_{x+t} is the probability that an individual aged $x+t$ will die in one year, and E_Q denotes the expectation under the risk-adjusted measure Q.

2.7. Insurance Premiums

In order to protect lenders from possible losses if non-repayment occurs, as well as to guarantee borrowers receiving monthly payments if lenders default on the loans, HUD provides mortgage insurance for the HECM program. Two insurance options are available for lenders to choose from, which are the assignment option and the shared premium option. However, none of the lenders choose the shared premium option because Fannie Mae does not purchase these loans. With the assignment option, FHA collects all the insurance premiums and the lender is allowed to assign the loan to FHA when the loan balance equals the adjusted property value. FHA takes over the HECM loan and pays an insurance claim to the lender covering his/her losses. By choosing this option, lenders are effectively shifting the collateral risk to HUD.

The mortgage insurance premiums (MIP) are paid by borrowers and include an upfront premium of 2% of the adjusted property value and an annual rate of 0.5% of the loan outstanding balance as long as the loan is active. Mathematically, the insurance premiums can be calculated as

$$\text{Premium} = 0.02H_0 + \sum_{t=1}^{\omega-x-1} {}_t p_x e^{-rt}\left(0.005L_t\right). \tag{3}$$

3. Major Risks in the HECM Loans

Reverse mortgages differ from traditional forward mortgages in the way that the outstanding loan balance grows due to principal advances, interest accruals, and other loan charges over the life of the loan. The loan balance may grow to exceed the property value at the time of termination because of multiple risks.

3.1. The Termination Risk

If a borrower lives a longer time than the expected lifespan, the principal advances and interest accruals will continue, which may lead the loan balance above the sale proceed of the property. In other words, lenders of reverse mortgages are faced with longevity risk. The mobility rate has the same effect on reverse mortgage products. Borrowers may move out of their homes because of their health conditions, marriage, divorce, death of the spouse, disasters, or simply the desire to live in another place. The mobility rate for the U.S. population is observed to decrease with age initially, but starts to increase after a certain old age such as 80 (Zhai, 2000). However, there is little data for us to calibrate the mobility rate for HECM borrowers. HUD simply assumes that the mobility rates are approximately 30 percent of the mortality rates in the HECM model (see, e.g., Rodda, Lam and Youn, 2004; Deutsche Bank Report, 2007). In other words, the termination rates are roughly 1.3 times the death rates. We use the same assumption in this study.

3.2. The Interest Rate risk

HECM loans almost exclusively opt for adjustable interest rates, therefore the variation of interest rates imposes additional uncertainty on HECM insurers. The rise of interest rates can result in a higher rate of interest accruals on the loan balance than anticipated, which increases the possibility of non-repayment when the loan eventually terminates. In this study, we choose a fixed interest rate with a risk adjustment, as most of the HECM models did. The question is: what is the appropriate risk premium in order to equate the value of the non-recourse provision to the present value of insurance premiums? We will answer this question in section 6.

3.3. The House Price Depreciation Risk

The uncertainty in house price depreciation rates is another risk we need to consider. If the house price remains stagnant or grows at a lower rate than anticipated, the outstanding loan balance at maturity may exceed the sale proceed of the property. Lenders may suffer from the losses and file insurance claims with the FHA. The house price depreciation risk is only partially diversifiable, because pooling mortgage products nationally only minimizes the risk of regional economic recession, but cannot diversify the risk of national economic recession. Additionally, the house price index is not a stationary time-series variable, which implies a simple risk adjustment is not applicable (Szymanoski, 1994).

4. Modeling the Longevity Risk

The traditional HECM model uses period life tables and therefore fails to capture the dynamics of longevity risk over time. In addition, it does not model the mortality improvement jumps explicitly. In the HECM program, an unexpected mortality improvement will increase the life expectancy and affect the premium pricing persistently. For example, heart disease is one of the leading causes of death in the United States. It accounts for 28.5 percent of the total death in 2002 according to the report of National Center of Health Statistics (Vol. 53, No. 5, October 2004). If there were a breakthrough of medicine protecting people from heart attack, the mortality rates would decrease substantially and the effects of mortality improvement would last for a long period, even forever. Therefore, we model longevity risk by incorporating permanent-effect jumps into the Lee-Carter model in this section.

4.1. The Lee-Carter Model Overview

Ever since Lee and Carter presented their original work in 1992, the Lee-Carter model has been widely used in mortality trend fitting and projection. The Census Bureau population forecast has used it as a benchmark for the long-run forecast of U.S. life expectancy. The two most recent Social Security Technical Advisory Panels have suggested the Trustees to adopt this method or other methods consistent with it (Lee and Miller, 2001).

Let $m_{x,t}$ denote the central death rate for age x at time t. The model decomposes this time series of age-specific death rates into two sets of age-specific constants a_x and b_x, and a time-varying factor k_t (Lee and Carter referred to k_t as mortality index. In order to distinguish k_t and the mortality index defined in the securitization contract, we refer to k_t as mortality factor thereafter). Mathematically, the Lee-Carter model can be represented as follows:

$$\ln(m_{x,t}) = a_x + b_x k_t + e_{x,t}, \tag{4}$$

where a_x represents the age pattern of death rates, b_x represents age-specific reactions to the time-varying factor, and $e_{x,t}$ is the error term which captures the age-specific effects not reflected in the model.

The Lee-Carter model cannot be fitted by the ordinary least square approach, because all variables on the right side of the model are unobservable. Moreover, this model is obviously over-parameterized. To obtain a unique solution, a normalization conditions is imposed such that the b_x terms sum to unity and the k_t terms sum to zero, i.e.,

$$\sum_x b_x = 1 \text{ and } \sum_t k_t = 0 .^4 \qquad (5)$$

Then a_x becomes the average value of $\ln(m_{x,t})$ over time, i.e.,

$$a_x = \frac{1}{T}\sum_{t=1}^{T}\ln(m_{x,t}) , \qquad (6)$$

where T is the length of the time series of mortality data.

Lee and Carter suggest a two-stage procedure to solve this problem. In the first stage, the singular value decomposition (SVD) method is applied to the matrix of $\ln(m_{x,t}) - a_x$ to obtain estimates of b_x, and k_t. In the second stage, the k_t factors are re-estimated by iteration, given the values of a_x and b_x obtained in the first step, such that the implied number of deaths equals to the actual number of deaths.

$$D_t = \sum_x \left(Pop_{x,t} \exp(a_x + b_x k_t)\right), \qquad (7)$$

where D_t is the actual total number of deaths at time t, and $Pop_{x,t}$ is the population in age group x at time t.

4.2. Model k_t with permanent-effect jumps

To make forecast of the future distribution of k_t, we need to choose a suitable model to fit k_t. Lee and Carter (1992) find a random walk with drift describes k_t well, and they employ an intervention model to remove the influence of the 1918 influenza pandemic. Li and Chan (2007) implement outlier detection and adjustment to unveil the “true” model underlying the outlier-free mortality series. Lin and Cox (2008) argue that mortality jumps need be taken into account in

[4] The normalization conditions can be chosen differently, but the choice of conditions won’t affect the estimation of parameters, a_x, b_x and k_t, up to a linear transformation (see Lee and Carter, 1992). It is necessary to point out that these constraints are used only once for the identification purpose in this step, and there are not such s constraints when we choose a proper model to fit and forecast the k_t series.

mortality securitization in order to hedge extreme mortality risks. Chen and Cox (2007) model k_t with a jump-diffusion process with transitory jump effects, since most of mortality jumps are caused by catastrophic events and only have short-term impacts. In the case of longevity risk modeling, we agree that mortality models must incorporate jumps to capture the rare longevity events. However, the jump effect needs to be modeled persistently, as longevity risk has a long-run impact: once the mortality is improved, it continues to evolve from this new level.

Denote N_t the number of jumps occurring in year *t*. For the sake of simplicity, we suppose there is at most one jump event in each year, with the probability of jump p.[5] That is,

$$N = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{with probability } 1-p \end{cases}. \tag{8}$$

We assume the jump severity variable, Y, is identically independently distributed normal variables with mean m and standard deviation s, and Y is independent of the jump frequency variable N.

The mortality factor k_t can be modeled as a random walk with drift, together with permanent longevity jumps,

$$k_{t+1} = k_t + \mu - pm + \sigma Z_{t+1} + Y_{t+1} N_{t+1}, \tag{9}$$

where Z_t is a standard normal random variable, independent of Y and N.

The parameter estimates can be obtained via MLE and are reported in Table 1. Based on these parameter values, we forecast the mortality factor k_t ($t = 1,2,3,...$) in 1,000 paths according to equation (9) and the central death rates based on the Lee-Carter Model. We then construct the

[5] More generally, we can use a Poisson random variable to model the jump frequency *N*. However, it does not add new insights as to the comparison between permanent jump effects and transitory jump effects, and only complicate the mathematics.

dynamic complete life table for every year. That is, calculate $q_{y,t}$ ($t = 1,2,3,...$), the probability that an individual aged y will die in one year at time t (see Appendix A for details).

Table 1: Parameter Estimates for k_t series via MLE

Parameter	Estimate	Parameter	Estimate
μ	-0.2172	σ	0.3872
m	-0.3062	s	2.3133
p	0.0396		

5. Modeling the House Price Depreciation Risk

5.1. The House Price Index

The underlying asset of the HECM product is the mortgaged property value. However, the mortgaged property is infrequently traded in the market and we seldom have historical data on each individual mortgaged property, which leaves us some difficulties on the pricing issue. Fortunately, we can mitigate this problem by resorting to the house price indices that reflect changes in residential property in a particular geographical region, since what we really care about is house price returns. In this study, we choose the nationwide House Price Index (HPI) from 1975 to 2007. The HPI is published by the Office of Federal Housing Enterprise Oversight (OFHEO) on a quarterly basis, and measures average price changes on single-family residential properties. It is based on a modified version of the weighted, repeated sales methodology, which was first proposed by Bailey, Muth and Nourse (1963), and later extended by Case and Shiller (1987, 1989). Repeated transactions of single-family houses are reviewed so that for each property, at least two mortgages were originated and subsequently purchased by Freddie Mac or Fannie Mae since January 1975. The use of repeated sales on the same property helps to control for differences in the quality of the houses in the sample. For this reason, the HPI is a “constant quality" house price index. As of December 1995, there were over 6.9 million repeated transactions in the national sample. Because of the breadth of the sample, it provides more information than is available in other house price indices. For more questions and comparison with other house price indices, we refer interested readers to the new released 2007 Q4 House Price Index Report.[6]

5.2. Housing Price Dynamics Review

In recent years, numerous studies model the dynamics of house prices. The main analytical issues examined by these studies include testing housing market efficiency, pricing contingent claims embedded in mortgages and mortgage-backed securities, and establishing a hedging mechanism for house price volatility (Cho, 1996). The consensus in the literature is that the house value is arguably not a stationary series. Variation of individual property values around the average increases over time.

Cunningham and Hendershott (1984), Kau, Keenan and Muller (1993) model property values as a geometric Brownian motion. The HUD also adopts this model framework in its HECM program, and assumes that property values appreciate with a 4 percent mean and a 10 percent standard deviation (Quercia, 1997). Under this model setup, nonstationarity arises from the fact that the cumulative house price appreciation rate over time is normally distributed, with mean and standard deviation growing over time. In addition, the house price appreciation is a random walk and has no memory, which means that previous values do not help in the prediction of future values.

However, tests of efficient market hypothesis (EMH) in the real estate markets provide us with contradictory results. The history of the housing literature on this topic goes back only

[6] Office of Federal Housing Enterprise Oversight (OFHEO). 2008. 4Q 2007 House Price Index Report, available at http://www.ofheo.gov/media/pdf/4q07hpi.pdf

about 20 years to the studies by Hamilton and Schwab (1985) and Linneman (1986). Since then, numerous studies have tested the EMH in the housing markets in the United States and other countries. Case and Shiller (1989) reject the weak-form efficiency in the U.S. housing market by pointing out the positive autocorrelation effects in both the changes in house prices and after-tax excess returns. Hosio and Pesando (1991), Ito and Hirono (1993) investigate the Toronto and Tokyo housing markets, respectively, and get similar results. The Institute of Actuaries (2005) also finds that there are positive autocorrelations in the Nationwide House Price Index in the U.K. Autocorrelations in the house price index actually give the price series some memory and allows speculative price bubbles, as well as mean reversion, to occur (Szymanoski, 1994).

Therefore, it is natural to apply time-series analysis to the real estate market. Particularly, the development of ARCH (autoregressive conditional heteroskedasticity) models relaxes the assumption of constant error variance, which can nicely explain the increasing volatility of house price dynamics. ARCH models were introduced by Engle (1982) and generalized as GARCH (Generalized ARCH) by Bollerslev (1986). These models are widely used in various branches of econometrics, especially in financial time series analysis. Chinloy, Cho and Megbolugbe (1997), Nothaft, Gao and Wang (1995) apply GARCH models in analyzing house price volatilities.

5.3. ARMA(R,M) – GARCH(P,Q)

To developing a GARCH model, we have to consider two specifications—one for the conditional mean and one for the conditional variance.

The conditional mean model $ARMA(R,M)$ can be expressed as

$$Y_t = c + \sum_{i=1}^{R} \phi_i X_{t-i} + \sum_{j=1}^{M} \theta_j \varepsilon_{t-j} + \varepsilon_t, \text{[7]} \qquad (10)$$

[7] The eigen values $\{\lambda_i\}$ associated with the characteristic AR polynomial $\lambda^R - \phi_1 \lambda^{R-1} - \phi_2 \lambda^{R-2} - ... - \phi_R$ must lie inside the unit circle to ensure stationarity. Similarly, the eigen values associated with the characteristic MA

where R is the order of the autocorrelation terms, M the order of the moving average terms, ϕ_i the i^{th}-order autocorrelation coefficient, θ_j the j^{th}-order moving average coefficient, and ε_t the Gaussian innovations.

Let Φ_{t-1} be the information set containing all the information up to time $t-1$. Denote σ_t^2 the conditional variance of the innovations given Φ_{t-1}, i.e., $\sigma_t^2 = E[\varepsilon_t^2 \mid \Phi_{t-1}]$. The conditional variance model $GARCH(P,Q)$ for the innovations can be written as

$$\sigma_t^2 = d + \sum_{i=1}^{P} \alpha_i \sigma_{t-i}^2 + \sum_{j=1}^{Q} \beta_j \varepsilon_{t-j}^2 \text{ ,}^{8} \tag{11}$$

where P is the order of the GARCH terms, Q the order of the ARCH term, α_i the i^{th}-order GARCH coefficient, and β_j the j^{th}-order ARCH coefficient.

Let X_t be the time series of the quarterly HPI from 1975 Q1 to 2007 Q4. We transform the data into the log-arithmetical return series, $Y_t = \ln(X_t) - \ln(X_{t-1})$, in order to analyze the house price returns. From Figure 1, the log return series is obviously not stationary. We implement the Augmented Dickey-Fuller (ADF) test, as well as the Phillips-Perron (PP) test. Both the ADF statistic (-0.1975) and PP statistic (4.1293) are higher than the critical values at the significance level of 5%, which means that the tests fail to reject the null hypothesis of a unit root in the return series. Therefore, we need to difference the return series to account for the order of integration. The return series becomes stationary after taking the first difference (see Figure 2), and the unit root tests confirm this result (ADF=-10.36413, PP=-18.64109).

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polynomial $\lambda^M + \theta_1 \lambda^{M-1} + \theta_2 \lambda^{M-2} + ... + \theta_M$ must lie inside the unit circle to ensure invertibility.

[8] We require $k > 0$, $\alpha_i \geq 0$ for $i = 1,2,...,P$, $\beta_i \geq 0$ for $j = 1,2,...,Q$, $\sum_{i=1}^{P} \alpha_i + \sum_{j=1}^{Q} \beta_j < 1$.

Figure 1: The HPI Log Returns

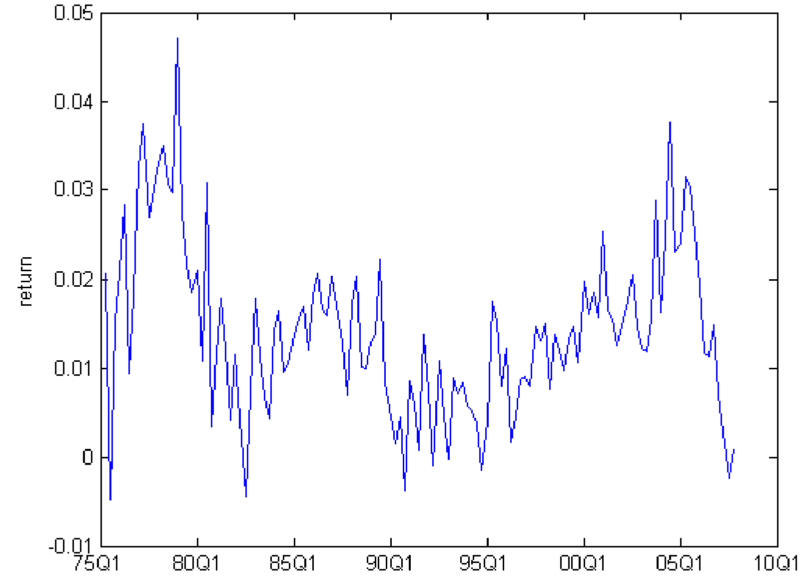
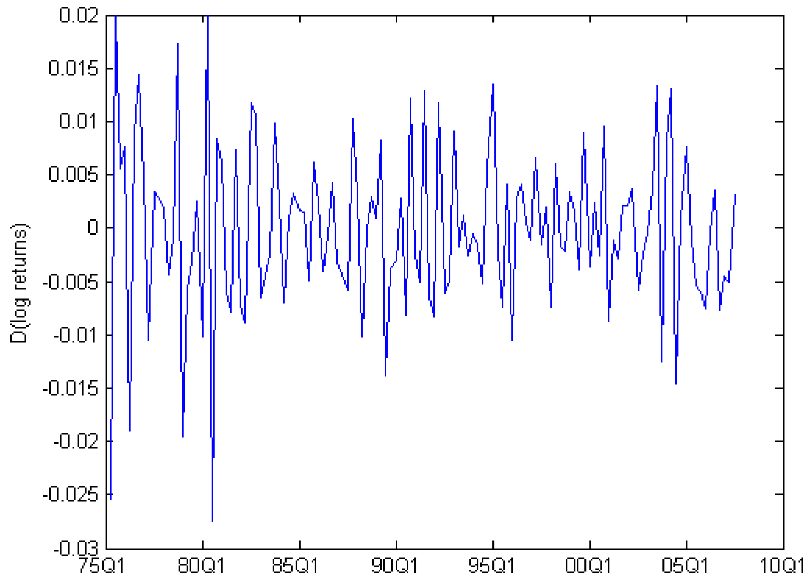


Figure 2: The First Difference of HPI Log Returns



The Box-Jenkins (1976) approach gives us a guide on how to identify an appropriate ARMA(R,M) model for the first difference of log returns (denoted as DY_t thereafter). We plot the sample autocorrelation coefficients (ACF) and partial autocorrelation coefficients (PACF) of DY_t in Figure 3 and 5, respectively. The sample ACFs are almost zero after one lag, and the sample PACFs die off after two lags. It seems that ARMA (2,1) would be appropriate for the DY_t series. However, after fitting DY_t with ARMA(2,1), we find that the constant term, AR(1) term, and MA(1) term are statistically insignificant at the 5% significance level. Therefore, more trials with other combinations of R and M are needed. The Akaike (AIC) and Bayesian (BIC) information criterion can be used as a guide for the appropriate lag order selection. After several trials, we find that ARMA (2,0) without the constant term makes a good fit. The coefficients of AR(1) and AR(2) terms are both highly significant. In addition, the AIC and BIC reach the lowest values (see Table 2).

Figure 3: ACFs of the First Difference of HPI Log Returns

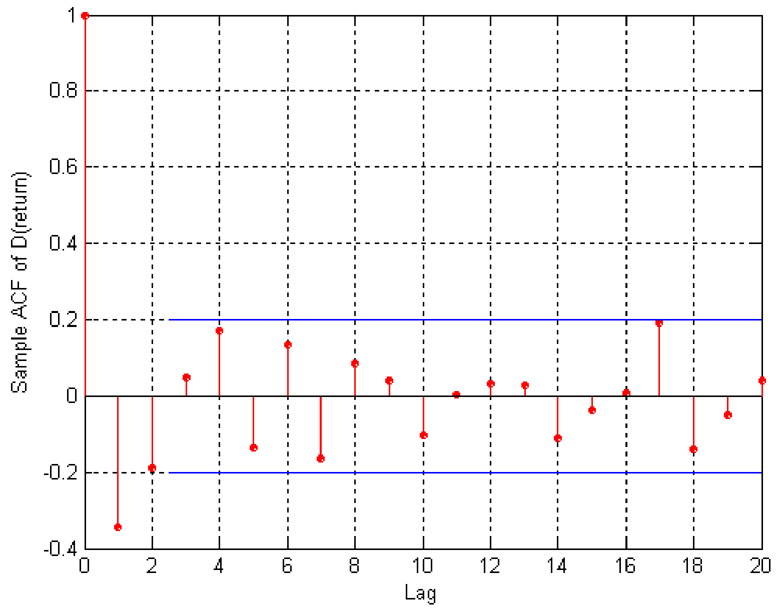


Figure 4: PACFs of the First Difference of HPI Log Returns

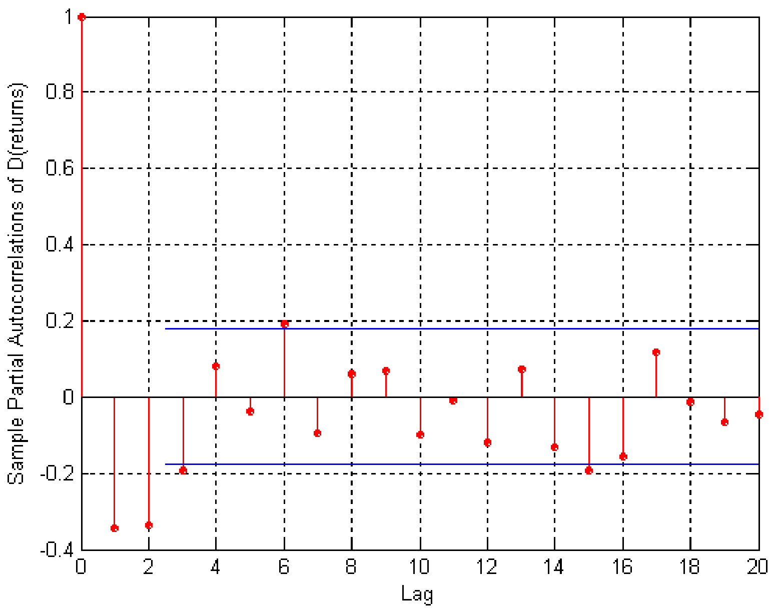


Table 2: Parameter Estimates of Several ARMA Models

Parameter	Value	Standard Error	T-statistic
ARMA(2,0) without constant term			
AR(1)	0.46137	-0.068178	-6.7672
AR(2)	-0.34567	0.074713	-4.6267
AIC	-911.8226	BIC	-903.2200
ARMA (2,0)			
C	-0.00028119	0.00063945	-0.4397
AR(1)	-0.46248	0.069209	-6.6824
AR(2)	-0.34666	0.074457	-4.6559
AIC	-910.0270	BIC	-898.5569
ARMA(2,1)			
C	-0.0001719	0.00043235	-0.3976
AR(1)	-0.16342	0.17813	-0.9174
AR(2)	-0.25641	0.10365	-2.4739
MA(1)	-0.35463	0.18916	-1.8747
AIC	-911.7503	BIC	-897.4126

After fitting a candidate ARMA specification, we should verify that there are no remaining autocorrelations that the model has not accounted for. Figure 5 plots the autocorrelations of innovations in the selected model. Almost all the ACFs are within the 95% confidence interval, which indicates that ARMA(2,0) without the constant term captures most of the autocorrelation effects in the DY_t series. We further verify this by implementing the Ljung-Box-Pierce Q-Test (see the upper panel of Table 3). When examined for up to 1, 5, and 10 lags of the ACFs at the 5% significance level, no significant correlation is present after fitting the DY_t series by ARMA(2,0) without a constant. We also plot the ACFs of the squared innovations in Figure 6, in order to examine the existence of ARCH effects. It demonstrates that, although the ACFs of the innovations exhibit little correlation, the ACFs of the squared innovations are still significantly correlated. The Ljung-Box-Pierce Q-Test on the squared residuals (the middle panel of Table 3) and Engle's ARCH test on the residuals (the lower panel of Table 3) show significant evidence in support of ARCH effects.

Figure 5: ACFs of Innovations of ARMA(2,0) without the Constant Term

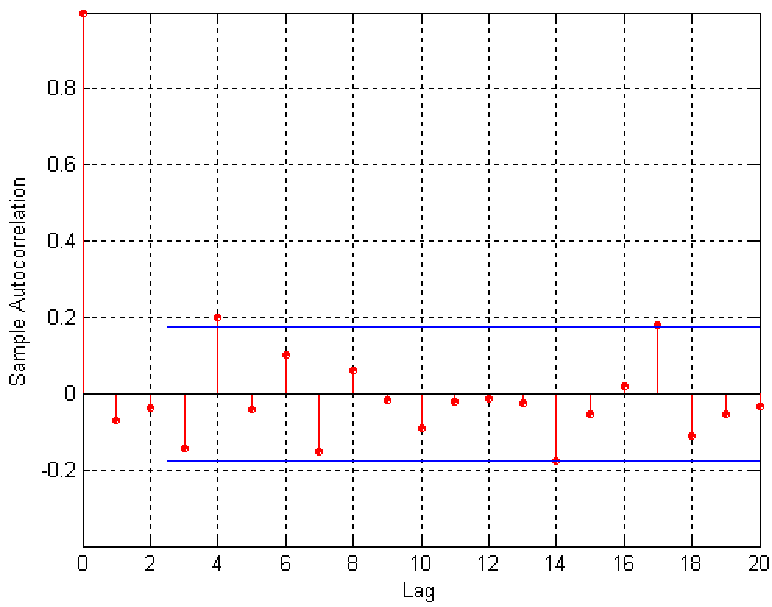


Figure 6: ACFs of Squared Innovations of ARMA(2,0) without the Constant Term

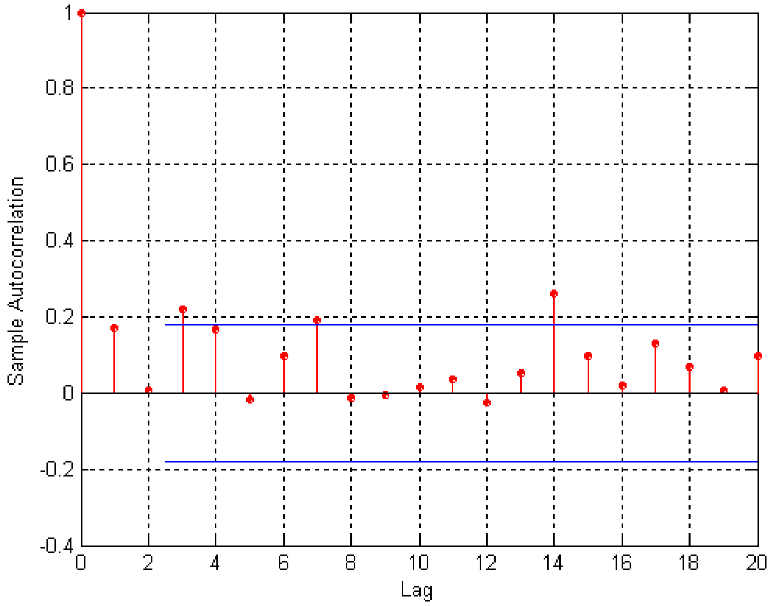


Table 3: Q-Test and ARCH Test for the ARMA(2,0) Model without the Constant Term

	Lag	Statistic	Critical Value	P-Value
Ljung-Box-Pierce Q-Test for the Innovations	1	0.6773	3.8415	0.4105
	5	9.2737	11.0705	0.0986
	10	15.6112	18.3070	0.1113
Ljung-Box-Pierce Q-Test for the Squared Innovations	1	3.9002	3.8415	0.0483
	5	14.1623	11.0705	0.0146
	10	20.5822	18.3070	0.0242
Engle's ARCH Test for the Innovations	1	5.9467	3.8415	0.0147
	5	10.1580	11.0705	0.0709
	10	11.2282	18.3070	0.3400

In most applications, GARCH(1,1) is enough to capture the ARCH effects. We keep ARMA(2,0) without the constant term for the conditional mean, and use GARCH(1,1) to model the conditional variance. Parameter estimates and corresponding statistics are reported in Table 4. All the coefficients are highly significant except for d, the constant term in the GARCH model.

Figure 7 shows that the standardized innovations (the innovations divided by the conditional standard deviations) become uncorrelated. In addition, almost all the ACFs of the squared standardized innovations fall in the 95 percent confidence interval (see Figure 8), which indicates that my model sufficiently explains the ARCH effects. The results of hypothesis testing in Table 5 confirm the qualitative checks above. The Q-test on the standardized innovations fails to reject the null hypothesis that there is no autocorrelation up to 10 lags. The Q-test on the squared standardized innovations and the ARCH test on the standardized innovations come to the same conclusion that conditional heteroskedasticity has disappeared after the model fitting.

Table 4: Parameter Estimates for the ARMA(2,0) – GARCH(1,1) model

Parameter	Value	Standard Error	T-statistic
AR(1)	-0.42727	0.083495	-5.1174
AR(2)	-0.34979	0.086188	-4.0584
d	4.0059e-006	3.3928e-006	1.1807
GARCH(1)	0.79897	0.12364	6.4620
ARCH(1)	0.10319	0.069692	1.4806

Figure 7: ACFs of Standardized Innovations of ARMA(2,0) + GARCH(1,1)

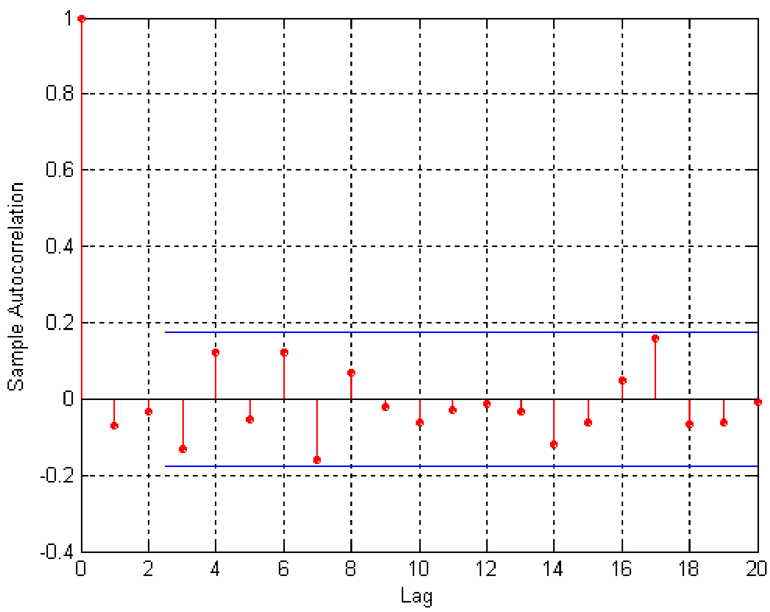


Figure 8: ACFs of Squared Standardized Innovations of ARMA(2,0) + GARCH(1,1)

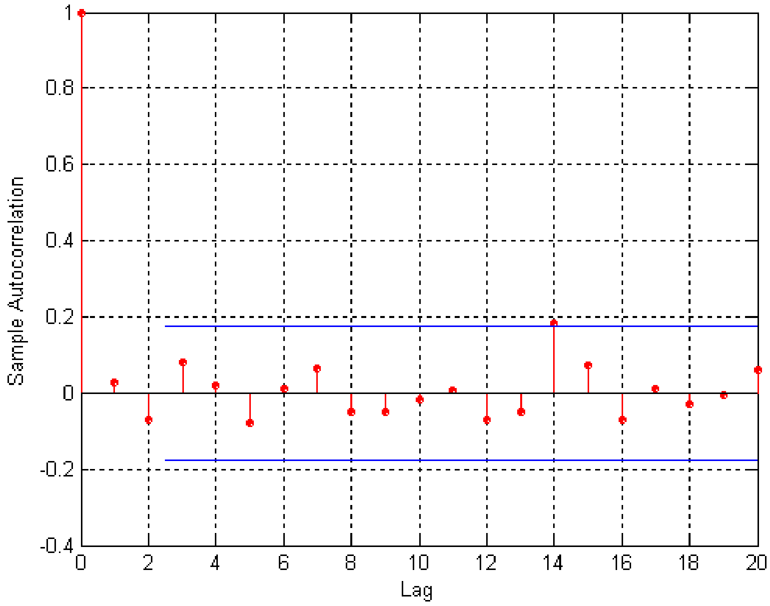


Table 5: Q-Test and ARCH Test for the ARMA(2,0) – GARCH(1,1) Model

	Lag	Statistic	Critical Value	P-Value
Ljung-Box-Pierce Q-Test for the Standardized Innovations	1	0.6668	3.8415	0.4142
	5	5.5319	11.0705	0.3545
	10	12.4285	18.3070	0.2574
Ljung-Box-Pierce Q-Test for the Squared Standardized Innovations	1	0.1049	3.8415	0.7461
	5	2.4678	11.0705	0.7813
	10	3.7699	18.3070	0.9571
Engle's ARCH Test for the Standardized Innovations	1	0.1639	3.8415	0.6856
	5	2.2089	11.0705	0.8196
	10	2.0212	18.3070	0.9962

6. The Pricing Framework

6.1. From Exponential Tilting to Conditional Esscher Transform

The exponential tilting of X with respect to the reference variable Y is defined as

$$f_X^*(x) = f_X(x)\frac{E[\exp(\lambda Y) \mid X = x]}{E[\exp(\lambda Y)]}, \tag{12}$$

where f_X and f_X^* represent the probability density function of X before and after the exponential tilting, respectively.

We leave much flexibility in the choice of reference variable Y. Particularly, if we choose the reference Y to be the risk X itself, the exponential tilting is reduced to the famous "Esscher transform" formula:

$$f_X^*(x) = f_X(x)\frac{E[\exp(\lambda X) \mid X = x]}{E[\exp(\lambda X)]} = f_X(x)\frac{\exp(\lambda x)}{E[\exp(\lambda X)]}. \tag{13}$$

The Esscher transform was introduced by Esscher (1932) and has become a time-honored tool in actuarial science. More recently, it has been applied to pricing financial and insurance securities in an incomplete market. Creating an equivalent martingale measure by the Esscher transform is justified by maximizing the expected power utility of an economic agent (Gerber and Shiu, 1994). See Pafumi (1997), Shiryaev (1999), Yao (2001), McLeish and Reesor (2003), and Yang (2004) for a rigorous theoretical background and more discussions.

Buhlmann, Delbaen, Embrechts and Shiryaev (1996) generalize the Esscher transform to stochastic processes and introduce the concept of the conditional Esscher transform. In terms of probability density functions, the conditional Esscher transform is defined as

$$f_{X_t}^*(x \mid \Phi_{t-1}) = f_{X_t}(x \mid \Phi_{t-1})\frac{\exp(\lambda_t x)}{E[\exp(\lambda_t X_t) \mid \Phi_{t-1}]}. \tag{14}$$

The pricing results under the conditional Esscher transform can be justified within the dynamic framework of utility maximization problems. Siu, Tong and Yang (2004) employ the conditional Esscher transform to price derivatives, assuming the underlying asset returns follow GARCH processes. Li, Boyle, Hardy and Tan (2007) follow the same line to price the No-

Negative-Equity-Guarantee in the U.K. equity release market, which is similar to the non-recourse provision of reverse mortgage programs in the U.S. We adapt their pricing framework when the underlying asset return follows an ARIMA-GARCH process.

6.2. Change of Measure Based on the Conditional Esscher Transform

Let (Ω, Φ, P) be a complete probability space, where P is the data-generating probability measure. Define two filtrations $F = (F_t)_{t \in \{0,1,,,,T\}}$ and $G = (G_t)_{t \in \{0,1,,,,T\}}$ on this probability space. The first filtration F is related to the development on the house price index, and F_t can be interpreted as the information obtained from observing the index up to time t. The second filtration G contains the information about the borrower of the revere mortgage, and G_t can be interpreted as the morality experience up to time t. In addition, the third filtration is defined as $\Phi = (\Phi_t)_{t \in \{0,1,,,,T\}}$, where $\Phi_t = F_t \vee G_t$. It means that Φ_t is the smallest σ-algebra that includes both F_t and G_t. This has the usual interpretation that, at time t, we have access to the combined information concerning both the development of the house price index and the mortality experience up to time t. Hence, Φ includes all the available information.

Let $T(x)$ denote the termination time of a reverse mortgage initiated to an individual aged x. For simplicity, assume $T(x)$ is stochastically independent of the housing price index, H_t, under the physical probability measure P. On the one hand, because the IPL is predetermined at the time of loan origination, the scheduled cash advances to borrowers are not reduced when property values fall. Smart borrowers will take advantage of this feature of reverse mortgages and won't have the incentive of refinancing during a recession. On the other hand, when property values rise, although borrowers have the incentive of refinancing, high release costs on the old mortgage and high closing costs on a new mortgage severely reduce the cash advances available under a new mortgage, which dampens borrowers' incentive of refinancing. See Lin and Tan (2003), Rodda, Lam and Youn (2004) for more discussions.

In section 5, we fit the first difference of house price returns, DY_t, using an ARMA(2,0)-GARCH(1,1) process, i.e.,

$$DY_t = \phi_1 DY_{t-1} + \phi_2 DY_{t-2} + \varepsilon_t ,$$

where $\varepsilon_t \mid \Phi_{t-1} \sim N(0, \sigma_t^2)$ and $\sigma_t^2 = d + \alpha_1 \sigma_{t-1}^2 + \beta_1 \varepsilon_{t-1}^2$.

Let $\mu_t = \phi_1 DY_{t-1} + \phi_2 DY_{t-2}$, then DY_t is normally distributed with mean μ_t and variance σ_t^2, given the information set Φ_{t-1}. That is, $DY_t \mid \Phi_{t-1} \sim N(\mu_t, \sigma_t^2)$. Note that μ_t and σ_t are not random given the information Φ_{t-1}. Consequently, the house price return series $Y_t \mid \Phi_{t-1} \sim N(\hat{\mu}_t, \sigma_t^2)$ under the physical measure P, where $\hat{\mu}_t = \mu_t + Y_{t-1}$.

Define a sequence $\{\Lambda_t\}_{t\in\{0,1,\ldots,T\}}$ with $\Lambda_0 = 1$, and for $t \ge 1$,

$$\Lambda_t = \prod_{k=1}^{t} \frac{\exp(\lambda_k Y_k)}{E[\exp(\lambda_k Y_k) \mid \Phi_{k-1}]}, \tag{15}$$

for some constants $\lambda_1, \lambda_2, \ldots, \lambda_t$. Buhlmann, Delbaen, Embrechts and Shiryaev (1996) prove that $\{\Lambda_t\}_{t\in\{0,1,\ldots,T\}}$ is a martingale.

Let P_t be the restriction of the measure P on the information Φ_t, where $P_T = P$. The fact $\{\Lambda_t\}_{t\in\{0,1,\ldots,T\}}$ is a martingale allows us to construct a family of measures $\{Q_t\}_{t\in\{1,2,\ldots,T\}}$ such that $dQ_t = \Lambda_t dP_t$, $Q_t = Q_{t+1} \mid \Phi_t$, and a probability measure $Q = Q_T$ on the sample space (Ω, Φ). See Siu, Tong and Yang (2004), Li, Boyle, Hardy and Tan (2007) for more details.

In order for Q to be a risk-neutral probability measure which is equivalent to the physical measure P on the same sample space (Ω, Φ), we choose a sequence of parameters

$\lambda_t^q = \dfrac{r-\mu_t}{\sigma_t^2} - \dfrac{1}{2}, (t = 1,2,...,T)$, such that the following set of equations are satisfied:

$$E_{Q_t}[\exp(Y_t); \lambda_t^q \mid \Phi_{t-1}] = \exp(r), \tag{16}$$

where r is the risk-free interest rate.[9]

Under the risk-adjusted measure Q, we have the following properties:

- $Y_t \mid \Phi_{t-1} \sim N\left(r - \dfrac{1}{2}\sigma_t^2, \sigma_t^2\right)$;
- $\Pr^Q(T(x) > t) = \Pr^P(T(x) > t) = {}_t p_x$;
- $T(x)$ is stochastically independent of H_t under Q;

The first property states exactly that the dynamics of Y_t under the risk-adjusted measure is the same as that under the physical measure, except that the mean is shifted by an amount of $-\hat{\mu}_t + r - \dfrac{1}{2}\sigma_t^2$ (see Appendix 4B for proof). The second property indicates the change of measure from P to Q does not affect the marginal distribution of the remaining lifetime of the reverse mortgage. Finally, the last property states that the independence between the termination time of loans and a priori given real estate market is preserved under Q. [10]

6.3. Numerical Illustrations

Recall the value of the non-recourse provision can be expressed as

$$\text{Value} = \sum_{t=0}^{\omega-x-1} E_Q\left[{}_t\, p_x q_{x+t} V_t\right], \tag{17}$$

where $V_t = e^{-rt}\max(L_t - H_t, 0)$ is the discounted payoff of an European exchange option

[9] We refer interested readers to Buhlmann et al. (1996) for a detailed proof.

[10] Property 2 and 3 arise from the fact that the Radon-Nikodym derivative, $\Lambda_t = dQ_t / dP_t$, used to obtain the probability measure Q is a function of Y_t, and hence is independent of $T(x)$.

matured at time t, exchanging H_t for L_t.

Suppose all home exits occur in the mid-year and let δ denote the average delay in time from the point of home exit until the actual sale of the property. Therefore, the time to maturity of the European exchange option is $t+0.5+\delta$. The value of the non-recourse provision becomes

$$\text{Value} = \sum_{t=0}^{\omega-x-1} E_Q\left[{}_t p_x q_{x+t} V_{t+0.5+\delta}\right]. \tag{18}$$

When the borrower dies in the future, the lender needs to sell the property and pay the transaction cost. Suppose the transaction cost is κ percent of the property value, then the discounted payoff of the European exchange option is

$$V_{t+0.5+\delta} = e^{-r(t+0.5+\delta)} \max(L_{t+0.5+\delta} - (1-\kappa)H_{t+0.5+\delta}, 0). \tag{19}$$

Further considering the rental yields from the property as dividends, we can adjust the option formula to accommodate rental yields. Assuming g is the rental yield per year, the discounted payoff of the European exchange option becomes

$$V_{t+0.5+\delta} = e^{-r(t+0.5+\delta)} \max\left(L_{t+0.5+\delta} - (1-\kappa)e^{-g(t+0.5+\delta)}H_{t+0.5+\delta}, 0\right). \tag{20}$$

At maturity, the house value is

$$H_{t+0.5+\delta} = H_0 e^{\sum_{i=1}^{4(t+0.5+\delta)} Y_i}, \tag{21}$$

and the loan balance is

$$L_{t+0.5+\delta} = \left(0.02H_0 + L_0\right)e^{u(t+0.5+\delta)}, \tag{22}$$

where u is the interest rate charged on the mortgage loan.[11]

Substituting equation (21) and (22) into equation (20), we obtain:

[11] In the HECM program, the loan balance increases with interest accruals and insurance premiums.

$$V_{t+0.5+\delta} = e^{-r(t+0.5+\delta)} \max\left(\left(0.02H_0 + L_0\right)e^{u(t+0.5+\delta)} - (1-\kappa)H_0 e^{-g(t+0.5+\delta)+\sum_{i=1}^{4(t+0.5+\delta)} Y_i}, 0 \right). \quad (23)$$

The key assumptions that we make for the numerical illustration are as follows.

- The average delay in time from the point of home exit until the actual sale of the property is six months, i.e., $\delta = 0.5$.
- The transaction cost of selling the house is 6 percent of the property value, i.e., $\kappa = 6\%$.
- In the HECM program, the risk-free interest rate is usually the 10 year U.S. Treasury rate. It is 3.91% per annum at 01/02/2008, which is equivalent to an interest rate of 3.84%, compounded continuously, i.e., $r = 3.84\%$.
- The interest rate charged on the loan is 4.71% per annum, which is equal to the one year constant maturity Treasury (CMT) rate of 2.71% plus a lender's spread margin of 150 bps and an additional HUD mortgage insurance premium of 50 bps. It is equivalent to 4.60% on a continuously compounding basis, i.e., $u = 4.6\%$.
- The rental yield is 2% per annum, compounded continuously, i.e., $g = 2\%$.
- The initial house value is assumed to be \$300,000, i.e., $H_0 = \$300,000$.
- We consider the lump sum payment option. Therefore, the loan amount L_0 is equal to the initial principle limit (IPL). IPL can be obtained using the online reverse mortgage calculator on Financial Freedom's website, assuming the property is located in Philadelphia, Zip code 19104. [12]

We first simulate the DY_t series for 1000 paths and transform DY_t to Y_t on each path. Applying the pricing framework discussed above, we change the probability measure from P to

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[12] Available at http://www.financialfreedom.com/calculator/Input_new.asp

Q on each path of Y_t, and then calculate the value of the non-recourse provision according to equation (18) and (23).

We also calculate the total insurance premiums collected by FHA for the purpose of comparison. Recall the insurance premiums consist of two parts, both paid by the borrowers: an up-front charge of 2% of the adjusted property value[13], and an annual rate of 0.5% of the outstanding loan balance for the life of the loan. Mathematically, it can be expressed as

$$\text{Premium} = 0.02H_0 + \sum_{t=1}^{\omega-x-1} {}_t p_x e^{-rt} \left(0.005 \times (0.02H_0 + L_0)e^{ut}\right). \tag{24}$$

The simulation results are shown in the panel of Scenario 1 in Table 7. Several observations here:

- Initial loan amount increases as the age at loan origination increases. This is reasonable because other conditions equal, the risk of reverse mortgages ultimately depends on when the maturity event occurs and for how long the loan has been accruing. The elder borrower has a shorter life expectancy, therefore the lender faces less longevity risk and can advance more cash amounts to the borrower.
- For the same reason, the value of the non-recourse provision decreases dramatically as the age at loan origination goes up. When the initial age is 62, the non-recourse provision is worth \$8,219. This figure drops to \$915 for a borrower at age 90.
- The insurance premiums also decrease with the age at loan origination. We can see that when the initial age increases, the borrower gets more cash advances, pays less insurance premiums, and can spend more money to improve the living standard of his

[13] The adjusted property value is the lesser of the appraised property value and the maximum loan limit. The maximum loan limit is placed by FHA depending on the geographical area of the borrower's house, taking into account the value of the house. For simplicity, we assume here the property value is always less than the maximum loan limit.

or her life.

- Theoretically, the value of the non-recourse provision should equal the insurance premiums the borrower pays to FHA in order to limit his/her liability to the property value. However, the present value of insurance premiums calculated based on current premium structure is far more than the actuarial present value of benefits. We report the ratio of the present value of insurance premiums to the value of non-recourse provision in Table 7. The ratio is 2.90 when the initial age is 62. It increases at an accelerating rate with the initial age of the borrower. When the borrower is 90 years old at the time of closing, the present value of insurance premiums collected by the FHA reaches 13.42 times the value of the non-recourse provision.

The U.S housing market has been experiencing the biggest slump recently. According to OFHEO Director James B. Lockhart, “The year 2007 showed the first four-quarter decline in the purchase-only index since its earliest data in 1991.” Prices fell between the third and fourth quarters of 2007 in every state except Maine. When compared with the data in the fourth quarter of 2006, house prices in the fourth quarter of 2007 depreciated by 6.6% in California, 5.9% in Nevada, and 4.7% in Florida (OFHEO, 2008). [14] It is interesting to ask what would happen if the value of the property had a sudden decrease at time of loan origination.

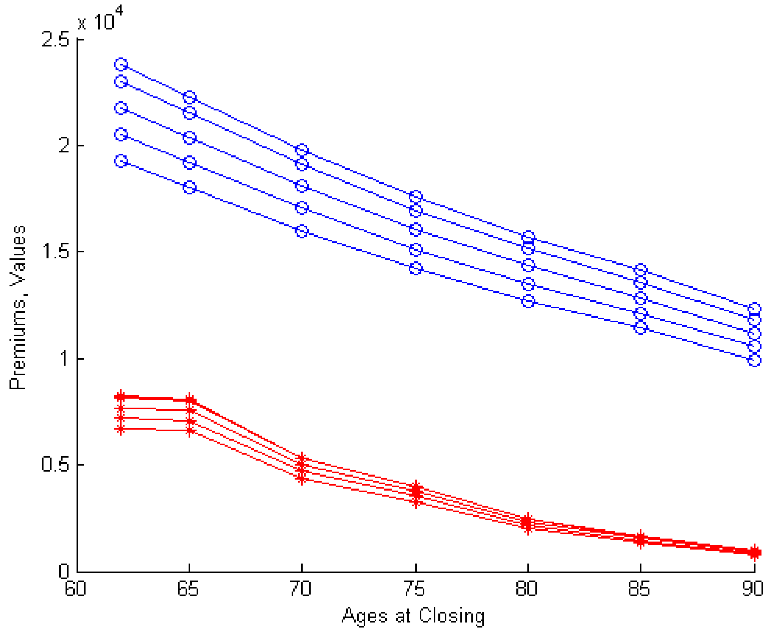
We analyze the effects of initial property values on the values of non-recourse provision and insurance premiums under different scenarios, assuming that the house price depreciates by 5%, 10%, 15% and 20%, respectively, at the time of closing. We predict that the present value of insurance premiums should decrease with the initial property value. That is because the house price depreciation at time of closing brings down the initial loan amount the borrower can obtain. The present value of insurance premiums decreases consequently, as the premium in each period

[14] OFHEO. 2008. 4Q 2007 House Price Index Report, available at http://www.ofheo.gov/media/pdf/4q07hpi.pdf

Table 6: Values of the Non-recourse Provision and Insurance Premiums in Different Scenarios

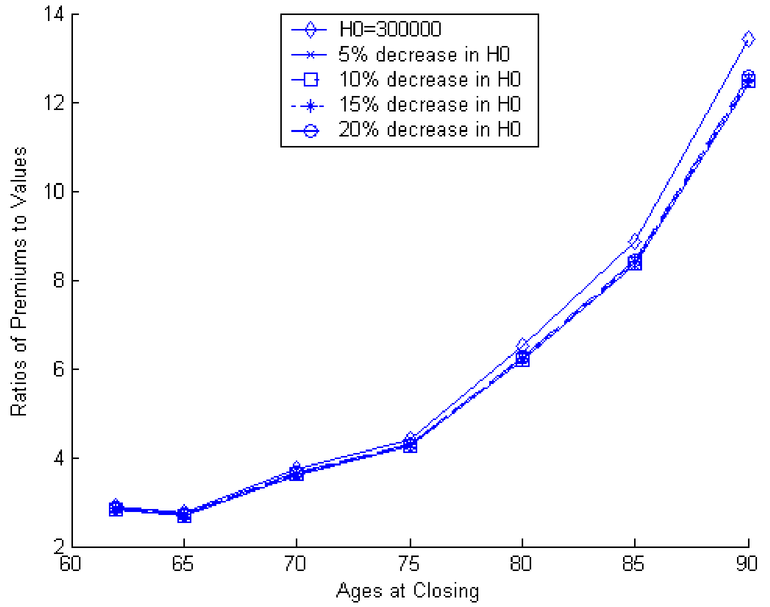
Age at Origination	62	65	70	75	80	85	90
Scenario 1: Initial Property Value H_0 = $300,000							
L_0	161,293	168,470	180,498	193,513	206,964	220,316	233,047
Value	8,219	8,071	5,319	3,967	2,414	1,592	915
Premium	23,811	22,286	19,796	17,550	15,713	14,110	12,282
Premium/Value	2.90	2.76	3.72	4.42	6.51	8.87	13.42
Scenario 2: Initial Property Value Decreases by 5%. H_0 = $285,000							
L_0	156,835	163,828	175,549	188,233	201,347	214,368	226,791
Value	8,125	7,990	5,292	3,973	2,448	1,631	952
Premium	23,004	21,525	19,107	16,927	15,144	13,586	11,809
Premium/Value	2.83	2.69	3.61	4.26	6.19	8.33	12.40
Scenario 3: Initial Property Value Decreases by 10%. H_0 = $270,000							
L_0	148,135	154,768	165,889	177,928	190,382	202,758	214,581
Value	7,658	7,532	4,989	3,745	2,307	1,537	898
Premium	21,746	20,351	18,071	16,013	14,330	12,859	11,181
Premium/Value	2.84	2.70	3.62	4.28	6.21	8.37	12.46
Scenario 4: Initial Property Value Decreases by 15%. H_0 = $255,000							
L_0	139,435	145,708	156,229	167,628	179,417	191,148	202,371
Value	7,191	7,074	4,685	3,518	2,166	1,444	843
Premium	20,488	19,177	17,034	15,100	13,516	12,133	10,552
Premium/Value	2.85	2.71	3.64	4.29	6.24	8.41	12.52
Scenario 5: Initial Property Value Decreases by 20%. H_0 = $240,000							
L_0	130,735	136,648	146,569	157,318	168,452	179,538	190,161
Value	6,724	6,616	4,382	3,289	2,025	1,350	789
Premium	19,230	18,004	15,997	14,185	12,702	11,406	9,923
Premium/Value	2.86	2.72	3.65	4.31	6.27	8.45	12.58

Figure 9: Values of the Non-recourse Provision and Insurance Premiums in Different Scenarios



Note: We graph 5 scenarios with the initial property value equal to $300,000, 285,000, 270,000, 255,000 and 240,000, respectively. The -o- line denotes present values of insurance premiums at different ages. It decreases with the initial property value. The -*- lines denotes the values of the non-recourse provision at different ages. It decreases with the initial property value, too.

Figure 10: Ratios of Insurance Premiums to Values of the Non-recourse Provision in Different Scenarios



is proportional to the outstanding loan balance. We predict the value of non-recourse provision should decrease with the initial property value, too, since it is simply the counterpart of insurance premiums and should behave similarly in response to the house price depreciation. Figure 9 confirms my predictions. However, it is noteworthy that the house price depreciation has little effect on the ratios of insurance premiums to the values of non-recourse provision (Figure 10).

In the above analysis, we assume the lender's margin is 150 bps. Here comes the next question: what is the actuarially fair risk premium that the lender should assess? In other words, what is the lender's margin that makes the present value of insurance premiums equal to the value of the non-recourse provision? Obviously, it depends on the initial age of the borrowers. We use Broyden's method to find out the actuarially fair risk premium and report it in Table 7. The finding is that the lender can achieve a much higher margin based on the current HECM insurance premium structure. The margin increases from 442 bps to 885 bps when the borrower's initial age changes from 62 to 90. It is explained by the fact that FHA charges more insurance premiums than the actuarial present value of claim payments. The intrinsic risk implied by high insurance premiums is correspondingly high. As a result, the lender can charge a higher risk margin on the loan balance.

Table 7: The Lender's Margin When H_0 = $300,000

Age at Origination	62	65	70	75	80	85	90
Lender's Margin (bps)	442	437	513	560	646	768	885

Note: Assume the initial property value equals $300,000, located in Philadelphia, Zip code 19104.

7. Conclusions and Discussions

The market for reverse mortgage products has started growing rapidly as an effective solution to the "Home Rich and Cash Poor" dilemma. As pointed out in the Deutsche Bank

Report (2007), “A larger number of originators, increased interest in securitization, and the aging population will lead to a more competitive and efficient origination.” The HECM program has favorable features and provides protection to both borrowers and lenders. On the one hand, borrowers benefit from the non-recourse provision which limits borrowers’ liability to the mortgaged property value. On the other hand, lenders are protected from the possible losses by the FHA insurance program. In this study, we model the various risks embedded in the HECM program and create an equivalent martingale measure to price the non-recourse provision. We further compare the value of the non-recourse provision with the present value of insurance premiums. We find that the premiums of HECM loans are adequate to cover expected claims. The FHA makes profits on the expected value basis.

The complexity of valuation problems comes from the fact that the HECM products are involved with multiple risks. We analyze longevity risk, mobility risk, and house price depreciation risk in this study. However, there are other risks we need to take into account, for example, basis risk, interest rate risk, and refinancing risk. First, the heart of pricing the non-recourse provision lies on modeling the dynamics of the underlying asset: the property value. We use the House Price Index to model house price returns, since each mortgaged property is infrequently traded in the market and does not have enough historical data available for us to analyze. Basis risk arises because the fluctuation of individual mortgaged property value may deviate from the HPI dynamics. Second, we assume a constant interest rate plus a lender’s margin for the life of the HECM loans so that we can determine the conditional Esscher parameters and create the risk-adjusted probability measure for pricing purposes. However, we do not live in a simple world with a flat term structure subject only to additive shifts (Boehm and Ehrhardt, 1994). Therefore, we do need to model the stochastic interest rates with a more realistic term structure, for instance, the Vasicek model (Vasicek, 1977) or CIR model (Cox, Ingersoll and Ross, 1985). Finally, the termination time of a reverse mortgage should be determined by a multiple-decrement model, considering mortality risk, mobility risk and refinancing risk. Similar to the refinancing of forward mortgages, reverse mortgages become refinanceable when interest rates decline or house prices increase. We assume that the refinancing rate is zero in this study. However, as the reverse mortgage market expands and matures, we will see higher refinancing rates. Further research based on a multiple decrement termination model is warranted.

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Appendix A: Construction of the Dynamic Complete Life Table

We have 11 age groups:

$g_0 = \{0\}$,
$g_1 = \{1,2,3,4\}$,
$g_2 = \{5,6,...,14\}$,
......,
$g_{10} = \{75,76,...,84\}$,
$g_{11} = \{85,86,...\}$

The question is: given $m_{k,t}$, the central-death-rates for the age group g_k at time t, how to obtain $q_{y,t}$, the probability that an individual aged y will die in one year at time t?

I use the following notations here (for simplicity, we omit the subscript t).

ω: The highest attainable age. Suppose ω = 110;

m_k **:** Central-death-rate for the age group g_k;

x_k: The first age in the age group g_k;

n_k : $x_{k+1} - x_k$;

l_y: Expected number of survivors to age y;

d_y: Expected number of deaths between ages y and $y+1$;

${}_nL_y$: The total expected number of years lived between ages y and $y+n$;

q_y: Probability that an individual aged y will die in one year.

Assuming that the function l_y is linear over the interval $x_k \le y \le x_{k+1}$, we have:

$$l_y = l_{x_k} - \frac{y - x_k}{n_k}(l_{x_k} - l_{x_{k+1}}), \qquad \text{(A.1)}$$

and

$$d_y = \frac{l_{x_k} - l_{x_{k+1}}}{n_k} = d_{x_k}\,, \tag{A.2}$$

for all $y \in g_k$

Now we need to use the central death rate m_k to determine q_{x_k} .

$${}_{n_k}L_{x_k} = \int_{x_k}^{x_{k+1}} l_y dy = \int_{x_k}^{x_{k+1}} \left(l_{x_k} - \frac{y - x_k}{n_k}(l_{x_k} - l_{x_{k+1}}) \right) dy\,,$$

$$= n_k l_{x_k} - d_{x_k} \int_0^{n_k} t dt = n_k l_{x_k} - d_{x_k} n_k^{\,2} / 2 \ . \tag{A.3}$$

Then

$$m_k = \frac{l_{x_k} - l_{x_{k+1}}}{{}_{n_k}L_{x_k}} = \frac{n_k d_{x_k}}{n_k l_{x_k} - d_{x_k} n_k^{\,2} / 2} = \frac{q_{x_k}}{1 - n_k q_{x_k} / 2}\,. \tag{A.4}$$

Solving for q_{x_k} , we obtain $q_{x_k} = \dfrac{m_k}{1 + n_k m_k / 2}$

The probability q_y for age $y \in g_k$ is determined by q_{x_k} :

$$q_y = \frac{d_y}{l_y} = \frac{d_{x_k}}{l_{x_k} - \frac{y - x_k}{n_k}(l_{x_k} - l_{x_{k+1}})} = \frac{d_{x_k}}{l_{x_k} - (y - x_k) d_{x_k}} = \frac{q_{x_k}}{1 - (y - x_k) q_{x_k}} \tag{A.5}$$

Appendix B: Change of Measure via the Conditional Esscher Transform

Given $Y_t \mid \Phi_{t-1} \sim N\left(\hat{\mu}_t, \sigma_t^2\right)$ under P , prove $Y_t \mid \Phi_{t-1} \sim N\left(r - \frac{1}{2}\sigma_t^2, \sigma_t^2 \right)$ under Q via the conditional Esscher Transform.

Proof:

The conditional Esscher transform can be described as:

$$f_{Q_t}(y \mid \Phi_{t-1}) = f_{P_t}(y \mid \Phi_{t-1}) \frac{\exp(\lambda_t y)}{E\left[\exp(\lambda_t Y_t) \mid \Phi_{t-1}\right]} \tag{B.1}$$

It immediately follows that the moment generating function of Y_t given Φ_{t-1} under the measure Q is given by

$$E_{Q_t}\left[\exp(zY_t);\lambda_t \mid \Phi_{t-1}\right] = E_{P_t}\left[\exp(zY_t)\frac{\exp(\lambda_t Y_t)}{E_{P_t}\left[\exp(\lambda_t Y_t) \mid \Phi_{t-1}\right]} \mid \Phi_{t-1}\right]$$

$$= \frac{E_{P_t}\left[\exp((z+\lambda_t)Y_t) \mid \Phi_{t-1}\right]}{E_{P_t}\left[\exp(\lambda_t Y_t) \mid \Phi_{t-1}\right]}. \quad \text{(B.2)}$$

Because $Y_t \mid \Phi_{t-1} \sim N\left(\hat{\mu}_t, \sigma_t^2\right)$ under P, the moment generating function of Y_t given Φ_{t-1} under the measure P is given by

$$E_{P_t}[\exp(zY_t) \mid \Phi_{t-1}] = \exp\left(\mu_t z + \frac{1}{2}\sigma_t^2 z^2\right). \quad \text{(B.3)}$$

Substituting equation (B.3) into equation (B.2)), we obtain

$$E_{Q_t}\left[\exp(zY_t);\lambda_t \mid \Phi_{t-1}\right] = \frac{\exp\left(\mu_t(z+\lambda_t) + \frac{1}{2}\sigma_t^2 (z+\lambda_t)^2\right)}{\exp\left(\mu_t\lambda_t + \frac{1}{2}\sigma_t^2\lambda_t^2\right)}$$

$$= \exp\left((\mu_t + \sigma_t^2\lambda_t)z + \frac{1}{2}\sigma_t^2 z^2\right). \quad \text{(B.4)}$$

Therefore,

$$E_{Q_t}\left[\exp(Y_t);\lambda_t^q \mid \Phi_{t-1}\right] = \exp\left(\mu_t + \sigma_t^2\lambda_t^q + \frac{1}{2}\sigma_t^2\right). \quad \text{(B.5)}$$

In order for Q to be an equivalent martingale measure, we need to have:

$$E_{Q_t}\left[\exp(Y_t);\lambda_t^q \mid \Phi_{t-1}\right] = \exp(r). \quad \text{(B.6)}$$

Equating the above two equations, we can solve

$$\lambda_t^q = \frac{r-\mu_t}{\sigma_t^2} - \frac{1}{2}, (t = 1,2,\ldots,T). \quad \text{(B.7)}$$

Substituting equation (B.7) into (B.5), we obtain

$$E_{Q_t}\left[\exp(zY_t);\lambda_t^q \mid \Phi_{t-1}\right] = \exp\left(\left(r - \frac{1}{2}\sigma_t^2\right)z + \frac{1}{2}\sigma_t^2 z^2\right), \tag{B.8}$$

which means the random variable Y_t, given the information Φ_{t-1}, is normally distributed with mean $r - \frac{1}{2}\sigma_t^2$ and variance σ_t^2 under Q.