

# RISKY CORPORATE DEBT WITH FINITE MATURITY

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ABSTRACT. We find the optimal maturity of risky corporate debt. We work in a setting where the firm chooses the maturity structure of its risky corporate debt by selecting the speed of repayment with a sinking fund provision. The trade off for the firm when it selects its optimal debt maturity structure is flexibility benefits versus issuing costs. In a dynamic setting the debt holders and equity holders have a common interest in re-optimizing the firms capital structure as the conditions for the firm change. However, the usual debt free-rider problem makes it in-optimal for equity holders to reduce the amount of debt by partly paying back debt as the firm's conditions worsen. However, it is optimal to issue new debt (with lower seniority) when the firm's conditions improve compared to what they were last time the firm issued debt. Because of the debt free-rider problem, the only way the firm can reduce its amount of outstanding debt (in our model) is by letting its finite maturity debt mature. The higher the firm selects its speed of repayment, the more of its debt matures per unit of time. This gives the firm more flexibility, since it can always issue some more (junior) debt. This flexibility comes at a cost, since debt (re-)issuing is costly. We show that there is an optimal choice of maturity structure counter-balancing flexibility and issuing costs. An implication of our model is that, in general, the higher the volatility of the firm's earnings processes, the shorter the optimal maturity (for given issuing costs). In order to verify this result of our model we regress the duration of the corporate debt of CRSP firms on the volatility of these firms earnings, and we find indeed a significantly negative relation.

## 1. INTRODUCTION

To be written!

## 2. STATIC MODEL

A flow of EBIT (measured in \$ rates per unit of time) from a production technology is given as

$$d\xi_t = \xi_t \mu dt + \xi_t \sigma dW_t.$$

Assume  $\mu < r$ .  $r$  is the *after tax* riskless interests rate. We consider a firm which issues debt and equity at date zero. Initially (at date zero) we assume that the EBIT process starts at one,  $\xi_0 = 1$ . At this date the firm optimally issues debt with principal  $P$  and coupon rate  $c$ . (We will find  $P$  and  $c$  later so it will actually be optimal.) The debt is going to be retired at a fixed rate  $\lambda$  of the remaining outstanding principal. That is, at date  $t$ , a fraction

$$\int_0^t \lambda e^{-\lambda s} ds = 1 - e^{-\lambda t}$$

of the debt issued at date zero is retired and hence the fraction  $e^{-\lambda t}$  is still outstanding. With this setup the issued debt will have an average maturity of

$$\int_0^\infty \lambda e^{-\lambda s} s ds = \frac{1}{\lambda}.$$

The average remaining time to maturity of the remaining fraction of the outstanding debt at a given date,  $t$ , would actually also be  $\frac{1}{\lambda}$ . That is,

$$\frac{1}{e^{-\lambda t}} \int_t^\infty \lambda e^{-\lambda s} (s - t) ds = \frac{1}{\lambda}.$$

The owner of the debt will get the following stream of cash flows at date  $t$

$$(1 - \tau_i)ce^{-\lambda t}dt + \lambda e^{-\lambda t}Pdt = ((1 - \tau_i)c + \lambda P)e^{-\lambda t}dt.$$

Here we assume that coupons are taxed at the rate  $\tau_i$ . Moreover, we define the principal of the debt so that the firm will always issue new debt at par, and, hence, it is a natural assumption that repayment of principal is not taxed. The owner of the equity will get the following stream of cash flows at date  $t$

$$(1 - \tau_e)\xi_t dt - ((1 - \tau_e)c + \lambda P)e^{-\lambda t} dt.$$

That is, the firm pays out the entire EBIT to its claim holders. There is no retained earnings in the firm.  $\tau_e$  denotes the total effective tax rate paid on dividends<sup>1</sup> to the equity holders. This tax rates is used for both positive and negative dividend streams. That is, we assume a perfectly symmetric tax system where deficits immediately trigger a flow of subsidies from the Government. Note that repayment of principal does not give any tax benefit. We will denote the date  $t$  values of debt and equity as  $D(\xi_t, t)$  and  $E(\xi_t, t)$  when the current level of the EBIT is  $\xi_t$ .

If the firm would never go bankrupt the value of debt and equity, at a give date  $t$ , can be calculated as

$$\begin{aligned} E_{\text{nb}}(\xi_t, t) &= \mathbb{E}_t \left[ \int_t^\infty e^{-r(u-t)} \left( (1 - \tau_e)\xi_u - ((1 - \tau_e)c + \lambda P)e^{-\lambda u} \right) du \right] \\ &= (1 - \tau_e) \int_t^\infty e^{-r(u-t)} \mathbb{E}_t[\xi_u] du - e^{rt} ((1 - \tau_e)c + \lambda P) \int_t^\infty e^{-(r+\lambda)u} du \\ (1a) \quad &= (1 - \tau_e)\xi_t \int_t^\infty e^{-(r-\mu)(u-t)} du - e^{rt} ((1 - \tau_e)c + \lambda P) \frac{1}{r + \lambda} e^{-(r+\lambda)t} \\ &= \frac{(1 - \tau_e)\xi_t}{r - \mu} - \frac{e^{-\lambda t} ((1 - \tau_e)c + \lambda P)}{r + \lambda} \end{aligned}$$

and

$$\begin{aligned} D_{\text{nb}}(\xi_t, t) &= \mathbb{E}_t \left[ \int_t^\infty e^{-r(u-t)} \left( ((1 - \tau_i)c + \lambda P)e^{-\lambda u} \right) du \right] \\ (1b) \quad &= \frac{e^{-\lambda t} ((1 - \tau_i)c + \lambda P)}{r + \lambda}. \end{aligned}$$

However, equity has limited liability so when the EBIT becomes too low the equity holders find it optimal to stop paying the coupons and thereby force the debt holders to declare the firm bankrupt. In order to take his option part of the equity value into account, we consider equity and debt as contingent claims on the EBIT process,  $\xi$ . Both debt and equity have infinite maturity and cash flow streams of the form  $(a\xi_t + be^{-\lambda t})dt$ . For the debt  $a = 0$  and  $b = (1 - \tau_i)c + \lambda P$ , and for equity  $a = 1 - \tau_e$  and  $b = -(1 - \tau_e)c - \lambda P$ . We denote the value of a generic contingent claim with a cash flow stream of the form  $(a\xi_t + be^{-\lambda t})dt$  as  $F(\xi_t, t)$ . This denotes the value at date  $t$  when the current EBIT value is  $\xi_t$ . Using standard arguments (based on no arbitrage) the function  $F(\cdot, \cdot)$  must fulfill the fundamental PDE

$$(2) \quad \frac{1}{2}\sigma^2\xi_t^2 F_{11}(\xi_t, t) + \mu\xi_t F_1(\xi_t, t) + F_2(\xi_t, t) - rF(\xi_t, t) + a\xi_t + be^{-\lambda t} = 0.$$

<sup>1</sup> $\tau_e$  includes both the corporate tax rate paid by the firm itself and the dividend tax rate paid by the equity holders. That is  $\tau_e = \tau_c + (1 - \tau_c)\tau_d$ . Hence, there is a tax advantage of having debt as long as  $\tau_e > \tau_i$ .

In order to solve this PDE we will make two transformation. Firstly, we will use a scaled EBIT process,  $X$ , defined as

$$(3) \quad X_t = e^{\lambda t} \xi_t,$$

as state variable instead of  $\xi$ . Secondly, we consider a scaled value function,  $\mathcal{F}$ , defined as

$$(4) \quad \mathcal{F}(X_t, t) = e^{\lambda t} F(e^{-\lambda t} X_t, t) = e^{\lambda t} F(\xi_t, t).$$

The partial derivatives of  $F$  and  $\mathcal{F}$  are related as follows,

$$(5a) \quad \mathcal{F}_1(X_t, t) = F_1(e^{-\lambda t} X_t, t) = F_1(\xi_t, t),$$

$$(5b) \quad \mathcal{F}_{11}(X_t, t) = e^{-\lambda t} F_{11}(e^{-\lambda t} X_t, t) = e^{-\lambda t} F_{11}(\xi_t, t),$$

and

$$(5c) \quad \begin{aligned} \mathcal{F}_2(X_t, t) &= \lambda e^{\lambda t} F(e^{-\lambda t} X_t, t) - \lambda X_t F_1(e^{-\lambda t} X_t, t) + e^{\lambda t} F_2(e^{-\lambda t} X_t, t) \\ &= \lambda e^{\lambda t} F(\xi_t, t) - \lambda e^{\lambda t} \xi_t F_1(\xi_t, t) + e^{\lambda t} F_2(\xi_t, t). \end{aligned}$$

Alternatively, we can write the system of equations (5) as,

$$(6a) \quad F_1(\xi_t, t) = \mathcal{F}_1(X_t, t),$$

$$(6b) \quad F_{11}(\xi_t, t) = e^{\lambda t} \mathcal{F}_{11}(X_t, t),$$

and

$$(6c) \quad F_2(\xi_t, t) = e^{-\lambda t} \mathcal{F}_2(X_t, t) - \lambda e^{-\lambda t} \mathcal{F}(X_t, t) + \lambda \xi_t \mathcal{F}_1(X_t, t).$$

Substituting  $\mathcal{F}$  from equation (4) and its partial derivatives from equation (6) into the fundamental PDE (2), it becomes

$$(7) \quad \begin{aligned} \frac{1}{2} \sigma^2 \xi_t^2 e^{\lambda t} \mathcal{F}_{11}(X_t, t) + \mu \xi_t \mathcal{F}_1(X_t, t) + e^{-\lambda t} \mathcal{F}_2(X_t, t) - \lambda e^{-\lambda t} \mathcal{F}(X_t, t) + \lambda \xi_t \mathcal{F}_1(X_t, t) \\ - r e^{-\lambda t} \mathcal{F}(X_t, t) + a \xi_t + b e^{-\lambda t} = 0. \end{aligned}$$

Multiplying every term in equation (7) with  $e^{\lambda t}$  yields

$$\begin{aligned} \frac{1}{2} \sigma^2 \xi_t^2 e^{2\lambda t} \mathcal{F}_{11}(X_t, t) + \mu \xi_t e^{\lambda t} \mathcal{F}_1(X_t, t) + \mathcal{F}_2(X_t, t) - \lambda \mathcal{F}(X_t, t) + \lambda e^{\lambda t} \xi_t \mathcal{F}_1(X_t, t) \\ - r \mathcal{F}(X_t, t) + a \xi_t e^{\lambda t} + b = 0, \end{aligned}$$

and then substituting  $X_t = e^{\lambda t} \xi_t$  yields

$$(8) \quad \frac{1}{2} \sigma^2 X_t^2 \mathcal{F}_{11}(X_t, t) + \mu X_t \mathcal{F}_1(X_t, t) + \mathcal{F}_2(X_t, t) - \lambda \mathcal{F}(X_t, t) + \lambda X_t \mathcal{F}_1(X_t, t) - r \mathcal{F}(X_t, t) + a X_t + b = 0.$$

Finally, if we collect terms in equation (8), we get

$$(9) \quad \frac{1}{2} \sigma^2 X_t^2 \mathcal{F}_{11}(X_t, t) + (\mu + \lambda) X_t \mathcal{F}_1(X_t, t) + \mathcal{F}_2(X_t, t) - (r + \lambda) \mathcal{F}(X_t, t) + a X_t + b = 0.$$

The claims we are pricing have infinite maturity and  $a$  and  $b$  are not time dependent.<sup>2</sup> Hence, our claims will only be functions of the scaled EBIT process,  $X$ . That is, we can reduce our PDE (9) to the ODE

$$(10) \quad \frac{1}{2}\sigma^2 X^2 \mathcal{F}''(X) + (\mu + \lambda)X \mathcal{F}'(X) - (r + \lambda)\mathcal{F}(X) + aX + b = 0.$$

Moreover, according to Itô's lemma, the scaled EBIT process,  $X$ , follows the process

$$dX_t = X_t(\mu + \lambda)dt + X_t\sigma dW_t.$$

The general solution to the ODE (10) is

$$(11) \quad \mathcal{F}(X) = f_1 X^{x_1} + f_2 X^{x_2} + \frac{aX}{r - \mu} + \frac{b}{r + \lambda}.$$

Here  $x_1$  and  $x_2$  are given as the two roots of the quadratic function corresponding to the Euler ODE (10)

$$Q(x) = \frac{1}{2}\sigma^2 x(x - 1) + (\mu + \lambda)x - (r + \lambda).$$

That is,

$$x_1 = \frac{(\frac{1}{2}\sigma^2 - \mu - \lambda) + \sqrt{(\mu + \lambda - \frac{1}{2}\sigma^2)^2 + 2(r + \lambda)\sigma^2}}{\sigma^2} > 1$$

and

$$x_2 = \frac{(\frac{1}{2}\sigma^2 - \mu - \lambda) - \sqrt{(\mu + \lambda - \frac{1}{2}\sigma^2)^2 + 2(r + \lambda)\sigma^2}}{\sigma^2} < 0.$$

Hence, the general solution to the PDE (2) is

$$(12) \quad \begin{aligned} F(\xi, t) &= e^{-\lambda t} \mathcal{F}(e^{\lambda t} \xi_t) \\ &= f_1 e^{-\lambda t} (e^{\lambda t} \xi)^{x_1} + f_2 e^{-\lambda t} (e^{\lambda t} \xi_t)^{x_2} + \frac{a\xi}{r - \mu} + \frac{e^{-\lambda t} b}{r + \lambda} \\ &= f_1 e^{\lambda(x_1 - 1)t} \xi^{x_1} + f_2 e^{\lambda(x_2 - 1)t} \xi_t^{x_2} + \frac{a\xi}{r - \mu} + \frac{e^{-\lambda t} b}{r + \lambda}. \end{aligned}$$

Now we can return to our valuation of debt and equity. As explained before, the equity holders have limited liability so when the debt holders declare the firm bankrupt, they take over the EBIT process (after paying a fraction of its value,  $\alpha$ , in bankruptcy costs) and are able to optimally lever it again. Hence, we will have the following boundary conditions on debt and equity at the (time dependent) threshold EBIT level,  $\xi_t$ , when the equity holders (dynamically optimally) withhold the coupon flow.

$$(13a) \quad E(\xi_t, t) = 0,$$

$$(13b) \quad E_1(\xi_t, t) = 0,$$

and

$$(13c) \quad D(\xi_t, t) = (1 - \alpha)A \frac{\xi_t}{\xi_0}.$$

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<sup>2</sup>It is essential that also the boundary conditions are not dependent on time. We will show that this is the case below.

$A$  is the (so far undetermined) value of the EBIT process when the current level of EBIT is one. In equation (13c) we take advantage of the positive homogeneity of degree one property of the solution. That is, the new owner's problem of finding the optimal leverage of the firm (including the maturity structure of the debt) after bankruptcy is exactly identical to the problem the original owner had at date zero, when the EBIT level was one. Hence, the value of a newly optimally levered firm when the current EBIT value is  $\xi_t$  times what it was when the EBIT was one. That is, the value is  $A\frac{\xi_t}{\xi_0}$ . Finally, at date zero we have the following initial conditions.

$$(13d) \quad D(\xi_0, 0) = P\xi_0,$$

$$(13e) \quad E(\xi_0, 0) = A\xi_0 - P(1 - k)\xi_0.$$

Here  $k$  is the issuing cost of new debt and  $P$  is the principal of the debt. Since the debt level of the firm is decaying exponentially in time, it seems economically plausible, that the threshold EBIT level for bankruptcy will do the same. That is, we conjecture that  $\xi_t = e^{-\lambda t}X$ , for a given constant,  $X$ . That is, if we make the same transformation of the debt and equity values as we did for the generic  $F$  function in equation (4), that is,<sup>3</sup>

$$\mathcal{E}(X_t) = e^{\lambda t}E(e^{-\lambda t}X_t, t) = e^{\lambda t}E(\xi_t, t)$$

and

$$\mathcal{D}(X_t) = e^{\lambda t}D(e^{-\lambda t}X_t, t) = e^{\lambda t}D(\xi_t, t),$$

then the boundary conditions from equation (13) in the transformed version become

$$(14a) \quad \mathcal{E}(X) = 0,$$

$$(14b) \quad \mathcal{E}'(X) = 0,$$

$$(14c) \quad \mathcal{D}(X) = (1 - \alpha)AX,$$

$$(14d) \quad \mathcal{D}(1) = P,$$

and

$$(14e) \quad \mathcal{E}(1) = A - P(1 - k).$$

Now that we are convinced that the both the boundary level and the boundary conditions are not depending on time, we can write the solution of the transformed debt and equity values as, cf. equation (11),

$$\mathcal{E}(X) = e_1X^{x_1} + e_2X^{x_2} + \frac{(1 - \tau_e)X}{r - \mu} - \frac{(1 - \tau_e)c + \lambda P}{r + \lambda}$$

and

$$\mathcal{D}(X) = d_1X^{x_1} + d_2X^{x_2} + \frac{(1 - \tau_i)c + \lambda P}{r + \lambda}.$$

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<sup>3</sup>As we will see, the transformed debt and equity functions will not depend on calendar time.

Hence, we also have the non-transformed debt and equity values as, cf. equation (12),

$$(15a) \quad E(\xi, t) = e_1 e^{\lambda(x_1-1)t} \xi^{x_1} + e_2 e^{\lambda(x_2-1)t} \xi_t^{x_2} + \frac{(1-\tau_e)\xi}{r-\mu} - \frac{e^{-\lambda t}((1-\tau_e)c + \lambda P)}{r+\lambda}$$

and

$$(15b) \quad D(\xi, t) = d_1 e^{\lambda(x_1-1)t} \xi^{x_1} + d_2 e^{\lambda(x_2-1)t} \xi_t^{x_2} + \frac{e^{-\lambda t}((1-\tau_i)c + \lambda P)}{r+\lambda}.$$

If we compare the values of debt and equity from equation (15) to the values we derived assuming no bankruptcy from equation (1), we can note that the last two terms in the equity value and the last term in the debt value are exactly the values we derived assuming no bankruptcy. In fact, this is a general result. The no-bankruptcy value is a particular solution to the inhomogeneous PDE, (equation (2) in our case), and the two first terms are the general solution to the homogeneous version of the PDE. The solutions to the homogeneous version can be interpreted as the various imbedded options to default.<sup>4</sup>

We use the five boundary conditions from equation (14) to find the five constants of the model,  $e_2$ ,  $d_2$ ,  $X$ ,  $P$ , and  $A$ .<sup>5</sup> Finally, we will find the optimal capital structure of the firm (e.g., parameterized via the promised coupon rate,  $c$ , of the debt) and the optimal maturity structure of the debt (parameterized by  $\lambda$ ) by maximizing the total value of debt and equity. Clearly the owner of the production technology (the EBIT generating device), must determine the terms of the debt contract (i.e., coupon rate and maturity structure) before the debt can be sold on the market. Hence, the owner would take into account the proceeds from selling the debt and the (value of the) equity in designing the optimal capital structure of the firm. Therefore, the owner would maximize  $E(\xi_0, 0) + (1-k)D(\xi_0, 0) = A\xi_0$ . That is, the owner determines the optimal coupon of the debt,  $c$ , and the optimal maturity structure of the debt,  $\lambda$ , as

$$\arg \max_{c, \lambda \geq 0} A.$$

### 3. STATIC MODEL WITH STABLE DEBT

In the previous model the firm only issues debt at date zero, so as time goes the debt level of the firm declines (unless  $\lambda = 0$ ) and the firm gets less and less tax benefits of debt. So (at least in some cases) it may be optimal for the equity holders to re-issue new debt. The first approach in this direction would be to let the equity holders maintain a stable debt level. In this version of the model, which is the Leland (1998) model, the equity holders simply re-issue exactly as much debt as they retire at each instant in time. At date zero, the firm issued debt with a coupon of  $c$  and a principal of  $P$ . At each instant in time thereafter the equity holders will continuously issue debt with a coupon  $\lambda c dt$  and a principal  $\lambda P dt$ . Hence, the total outstanding principal at date  $t$  will be

$$P e^{-\lambda t} + \int_0^t \lambda P e^{-\lambda(t-s)} ds = P,$$

<sup>4</sup>To be concrete, blah, blah.

<sup>5</sup>The two remaining constants,  $e_1$  and  $d_1$  are zero, because that  $x_1 > 1$ . (We have the two boundary conditions  $\lim_{X \rightarrow \infty} \frac{E(X)}{X} < \infty$  and  $\lim_{X \rightarrow \infty} \frac{D(X)}{X} < \infty$ .)

and the total coupon at date  $t$  will be

$$ce^{-\lambda t} + \int_0^t \lambda ce^{-\lambda(t-s)} ds = c,$$

and the total debt retirement at date  $t$  (measured in \$ units, i.e., how much money is used to pay back principal) will be

$$\lambda Pe^{-\lambda t} dt + \int_0^t \lambda^2 Pe^{-\lambda(t-s)} ds dt = \lambda P dt.$$

The owner of the equity will get the following stream of cash flows at date  $t$

$$(1 - \tau_e)\xi_t dt - (1 - \tau_e)c dt + \lambda(D(\xi) - P) dt.$$

The last term originates because the debt that the firm issues after date zero is not (unless the EBIT value is identical to the value it had at date zero) issued at par. That is, the cash flow to the equity holders for retiring old debt and re-issuing new debt nets out exactly.

The owner of the debt issued at date  $s$  will get the following stream of cash flows at date  $t$ ,  $t \geq s$ ,

$$(1 - \tau_i)ce^{-\lambda(t-s)} dt + \lambda e^{-\lambda(t-s)} P dt = ((1 - \tau_i)c + \lambda P)e^{-\lambda(t-s)} dt.$$

Hence, if we add up all the streams of cash flows from all the debt issues, the total stream of cash flows is

$$((1 - \tau_i)c + \lambda P) \left( e^{-\lambda t} dt + \int_0^t e^{-\lambda(t-s)} ds dt \right) = ((1 - \tau_i)c + \lambda P) dt.$$

If we look at it as one person holding all the debt, then we need to subtract the outflow from buying the bonds, i.e., the money that the bond holder lends to the firm. If we subtract these outflows (after the initial outflow at date zero) we get

$$((1 - \tau_i)c + \lambda P) dt - \lambda P dt = (1 - \tau_i)c.$$

With this interpretation of the debt, both equity and debt has cash flow streams that are identical to the cash flow streams in the static model with  $\lambda = 0$ . Hence, values and bankruptcy barriers will be identical to the static model in the case when the average maturity is infinite ( $\lambda = 0$ ).

It is, however, interesting to look further into how the debt issued at different points in time is valued. Leland (1998) does not say anything about this part of his model. If all the issued debt would split the value equally in case of bankruptcy, then we can value each issue of debt by applying the debt issue weight on the total value of all the outstanding debt.

#### 4. DYNAMIC MODEL

In the previous model the equity holders maintains a fixed constant debt level. this is the debt level that maximizes the firm value at date zero, but it may not be the optimal debt level at the date of issue. If, e.g., at date  $t$  the EBIT level has dropped compared to what it was at date zero, then the optimal debt level at date  $t$  would be less than what was optimal at date zero, so why should the firm maintain a stable debt level?

In the dynamic version of the model, we allow the equity holders to re-issue debt continuously in time in an dynamically optimal way. The first thing we need to figure out is, when is it actually dynamically optimal for the equity holders to re-issue debt.

When the firm issues new debt the equity holders get the proceeds from selling the new debt. However, it is not clear what the proceeds are. We can think of it in two ways:

- (1) The firm issues subordinated debt. Standard debt covenants clauses in the debt contracts already issued indicate that the new issue is not supposed to leave the existing debt worse off. On the other hand, the equity holders does not have any incentive to leave the holders of the already issued debt better off, since this will just reduce the proceeds to the equity holders of any new issues. This leads to a fairly complicated debt contract about how to prioritize the different classes of debt in case of bankruptcy.
- (2) The firm buys back all the outstanding debt at its market value from the existing debt holders before it issues new debt.

We will assume (2). So when the firm is going to issue new debt, the new issue will be issued when there is no debt in the firm at all. Hence, if the firm re-issues debt at date  $t$ , when the EBIT level is  $\xi_t$ , then the optimal coupon of the new issue will be  $c\xi_t$  and the corresponding value of the debt (which is also the new principal, since we issue at par) will be  $P\xi_t$ .

Given the assumption (2), the market value of the currently outstanding debt is as if there would never be any further issues, since the equity holders always buy back the outstanding debt before issuing any new debt. Hence, the debt will never have any subordinated debt on top of it. That is, we can use the value from the static model.

Assume that the firm issued debt at the optimal level at date  $t_0$ . The debt issued at this date has a coupon of  $c\xi_{t_0}$  and a principal of  $P\xi_{t_0}$ . The value of this debt at date  $t$ ,  $t \geq t_0$ , is then

$$D\left(\frac{\xi_t}{\xi_{t_0}}, t - t_0\right)\xi_{t_0} = e^{-\lambda(t-t_0)}\mathcal{D}\left(\frac{X_t}{X_{t_0}}\right)\xi_{t_0} = e^{-\lambda t}\mathcal{D}\left(\frac{X_t}{X_{t_0}}\right)X_{t_0}.$$

Here we use the results from the static model. Hence, if the firm decides to issue new debt at date  $t$ , and the last debt was issued at date  $t_0$ , then the net proceeds of the issue will be

$$(1 - k)\left(P\xi_t - D\left(\frac{\xi_t}{\xi_{t_0}}, t - t_0\right)\xi_{t_0}\right) = (1 - k)e^{-\lambda t}\left(PX_t - \mathcal{D}\left(\frac{X_t}{X_{t_0}}\right)X_{t_0}\right).$$

Since  $\mathcal{D}$  is a strictly concave function and  $\mathcal{D}(1) = P$ , the net proceeds are strictly negative for  $X_t < X_{t_0}$  and strictly positive for  $X_t > X_{t_0}$ . Hence, it is optimal for the equity holders to re-issue new debt as soon as  $X_t$  is greater than  $X_{t_0}$ . (This assumes no fixed transactions cost for re-issuing debt.) We will assume that the equity holders re-issue debt gradually as

$$M_t = \max_{s \in [0, t]} X_s$$

increases. Hence, we can use a first order approximation in order to find the net proceeds from the debt issue

$$\begin{aligned} (1 - k)e^{-\lambda t}\left(PX_t - \mathcal{D}\left(\frac{X_t}{X_{t_0}}\right)X_{t_0}\right) &= (1 - k)e^{-\lambda t}\left(PX_t - \left(\mathcal{D}(1) + \mathcal{D}'(1)\left(\frac{X_t}{X_{t_0}} - 1\right)\right)X_{t_0}\right) \\ &= (1 - k)e^{-\lambda t}(PX_t - PX_{t_0} - \mathcal{D}'(1)(X_t - X_{t_0})) \\ &= (1 - k)e^{-\lambda t}(P - \mathcal{D}'(1))(X_t - X_{t_0}). \end{aligned}$$

That is, the equity holders will get the following stream of net proceeds for re-issuing debt

$$(1 - k)e^{-\lambda t}(P - \mathcal{D}'(1))dM_t.$$

In order to find out how much debt the firm has outstanding, we need the maximum of past EBITs with an exponential decaying part. We will use the state variable  $m$  to keep track of this

$$m_t = \max_{s \in [0, t]} \xi_s e^{-\lambda(t-s)}.$$

We also need to know the date when this maximum was obtained. We denote this date  $\nu_t$ . That is  $m_t = \xi_{\nu_t} e^{-\lambda(t-\nu_t)}$ .<sup>6</sup> At that date the principal was  $P\xi_{\nu_t}$  and the coupon rate was  $c\xi_{\nu_t}$ .

Date  $\nu_t$  was the last date (seen from date  $t$ ) when the firm (optimally) issues debt. (We will find  $P$  and  $c$  later so it will actually be optimal.) The debt is going to be retired at a rate  $\lambda$ . That is, at date  $t$ , a fraction

$$\int_{\nu_t}^t \lambda e^{-\lambda s} ds = 1 - e^{-\lambda(t-\nu_t)}$$

of the debt issued at date  $\nu_t$  is retired and hence the fraction  $e^{-\lambda(t-\nu_t)}$  is still outstanding. With this setup the issued debt will have an average maturity of

$$\int_{\nu_t}^{\infty} \lambda e^{-\lambda s} s ds = \frac{1}{\lambda}.$$

The owner of the latest issued debt (that is, the debt issued at date  $\nu_t$ ) will get the following cash flow at date  $t$

$$(1 - \tau_i)c\xi_{\nu_t} e^{-\lambda(t-\nu_t)} dt + \lambda e^{-\lambda(t-\nu_t)} P\xi_{\nu_t} dt = ((1 - \tau_i)c + \lambda P)m_t dt$$

$m_t$  is related to  $M_t$  in the following way

$$e^{\lambda t} m_t = \max_{s \in [0, t]} \xi_s e^{\lambda s} = \max_{s \in [0, t]} X_s = M_t.$$

Between points in time when the firm issues new debt, the equity holders will get the following stream of cash flows at date  $t$

$$(1 - \tau_e)\xi_t dt - ((1 - \tau_e)c + \lambda P)m_t dt = e^{-\lambda t} \left( (1 - \tau_e)X_t - ((1 - \tau_e)c + \lambda P)M_t \right) dt.$$

If we also take into account net proceeds of debt issues, the total stream of cash flows to equity at date  $t$  will be

$$e^{-\lambda t} \left( \left( (1 - \tau_e)X_t - ((1 - \tau_e)c + \lambda P)M_t \right) dt + (1 - k)(P - \mathcal{D}'(1))dM_t \right).$$

Similarly, the owner of the latest issued debt (that is, the debt issued at date  $\nu_s$ ) will get the following cash flow at date  $t$

$$((1 - \tau_i)c + \lambda P)m_t dt = e^{-\lambda t} ((1 - \tau_i)c + \lambda P)M_t dt.$$

In the dynamic model, we will again consider both debt and equity as contingent claims. In this case, however, the contingent claims will be a function on both the scaled EBIT process,  $X$ , and the past maximum of the scaled EBIT process,  $M$ . Both debt and equity are infinite maturity claims with cash flow stream of the form  $e^{-\lambda t}((aX_t + bM_t)dt + g dM_t)$ . For the debt  $a = 0$ ,  $b = ((1 - \tau_i)c + \lambda P)$ , and  $g = 0$ , and for equity

<sup>6</sup>The firm may have issued even more debt in the past (before date  $\nu_t$ ), but because of the sinking fund feature, part of this debt is now retired and we may even have issued debt after this date. The point is that date  $\nu_t$  is the latest date when we have issued debt before (and including) date  $t$ . E.g, if  $\lambda = \infty$ , then  $m_t = \xi_t$  and  $\nu_t = t$  always.

$a = 1 - \tau_e$ ,  $b = -((1 - \tau_e)c + \lambda P)$ , and  $g = (1 - k)(P - \mathcal{D}'(1))$ . As it was the case in the static model, the value of debt and equity under the assumption that the firm will never go bankrupt is a way to find a particular solution to the in-homogeneous PDE. We will therefore do the same in the dynamic model. We will derive the value of a general claim, denoted  $F_{\text{nb}}(X_t, M_t, t)$ , with a cash flow stream  $e^{-\lambda t}((aX_t + bM_t)dt + g dM_t)$  and assume that this stream will continue forever. In order to get rid of the time dependence of the cash flow stream, we can apply the same scaling of the value function as we did in the static model. That is, we value a general claim on the cash flow stream  $(aX_t + bM_t)dt + g dM_t$ . The value of this claim, which we will denote  $\mathcal{F}_{\text{nb}}(X, M)$ , will be related to  $F$  as follows

$$F_{\text{nb}}(X, M, t) = e^{-\lambda t} \mathcal{F}_{\text{nb}}(X, M).$$

If the firm would never go bankrupt, the value of the claim, at a give date  $t$ , can be calculated as

$$F_{\text{nb}}(X_t, M_t, t) = \mathbb{E}_t \left[ \int_t^\infty e^{-r(u-t)} e^{-\lambda u} ((aX_u - bM_u)du + g dM_u) \right].$$

Hence, the value of the scaled claim,  $\mathcal{F}_{\text{nb}}(X_t, M_t)$  at a give date  $t$  can be calculated as<sup>7</sup>

$$\begin{aligned} \mathcal{F}_{\text{nb}}(X_t, M_t) &= \mathbb{E}_t \left[ \int_t^\infty e^{-r(u-t)} e^{-\lambda(u-t)} ((aX_u - bM_u)du + g dM_u) \right] \\ &= \mathbb{E}_t \left[ \int_t^\infty e^{-(r+\lambda)(u-t)} ((aX_u - bM_u)du + g dM_u) \right] \\ &= a \int_t^\infty \mathbb{E}_t [e^{-(r+\lambda)(u-t)} X_u] du - b \int_t^\infty e^{-(r+\lambda)(u-t)} \mathbb{E}_t [M_u] du + g \mathbb{E}_t \left[ \int_t^\infty e^{-(r+\lambda)(u-t)} dM_u \right]. \end{aligned}$$

As in the static model we have

$$\mathbb{E}_t [e^{-(r+\lambda)(u-t)} X_u] = e^{-(r-\mu)(u-t)} X_t.$$

Moreover,

$$\begin{aligned} \mathbb{E}_t [M_u] &= M_t (1 - \Phi(e - \sigma\sqrt{u-t})) + e^{(\mu+\lambda)(u-t)} X_t \left( 1 + \frac{\sigma^2}{2(\mu+\lambda)} \right) \Phi(e) \\ &\quad - X_t \frac{\sigma^2}{2(\mu+\lambda)} \left( \frac{X_t}{M_t} \right)^{-\frac{2(\mu+\lambda)}{\sigma^2}} \Phi \left( e - \frac{2(\mu+\lambda)}{\sigma} \sqrt{u-t} \right). \end{aligned}$$

The last equation is the value (except for the discounting with  $e^{-(r+\lambda)(u-t)}$ ) of the European look-back call option when the fixed exercise price is zero. The constant  $e$  in this equation is

$$e = \frac{\ln \frac{X_t}{M_t} + \left( \mu + \lambda + \frac{\sigma^2}{2} \right) (u-t)}{\sigma\sqrt{u-t}}.$$

Again, as in the static model

$$\int_t^\infty \mathbb{E}_t [e^{-(r+\lambda)(u-t)} X_u] du = X_t e^{(r-\mu)t} \int_t^\infty e^{-(r-\mu)u} = \frac{X_t}{r-\mu}.$$

<sup>7</sup>As we will see,  $\mathcal{F}_{\text{nb}}$  will not depend on calendar time.

In order to find the value of the general claim if the firm never goes bankrupt, we need to derive

$$\int_t^\infty e^{-(r+\lambda)(u-t)} \mathbb{E}_t[M_u] du$$

and

$$\mathbb{E}_t \left[ \int_t^\infty e^{-(r+\lambda)(u-t)} dM_u \right]$$

Firstly, note that the two are related by the integration by parts formula

$$\int_t^\infty e^{-(r+\lambda)(u-t)} \mathbb{E}_t[M_u] du = \frac{1}{r+\lambda} \left( M_t + \mathbb{E}_t \left[ \int_t^\infty e^{-(r+\lambda)(u-t)} dM_u \right] \right).$$

Secondly, note that if we introduce

$$Z_t = \frac{M_t}{X_t}$$

then

$$\begin{aligned} \frac{\mathbb{E}_t[M_u]}{X_t} &= Z_t (1 - \Phi(e - \sigma\sqrt{u-t})) + e^{(\mu+\lambda)(u-t)} \left( 1 + \frac{\sigma^2}{2(\mu+\lambda)} \right) \Phi(e) \\ &\quad - \frac{\sigma^2}{2(\mu+\lambda)} Z_t^{\frac{2(\mu+\lambda)}{\sigma^2}} \Phi \left( e - \frac{2(\mu+\lambda)}{\sigma} \sqrt{u-t} \right) \end{aligned}$$

and  $e$  reduces to

$$e = \frac{-\ln(Z_t) + \left( \mu + \lambda + \frac{\sigma^2}{2} \right) (u-t)}{\sigma\sqrt{u-t}}.$$

Now, define

$$g(z, u) = z \left( 1 - \Phi(e - \sigma\sqrt{u}) \right) + e^{(\mu+\lambda)u} \left( 1 + \frac{\sigma^2}{2(\mu+\lambda)} \right) \Phi(e) - \frac{\sigma^2}{2(\mu+\lambda)} z^{\frac{2(\mu+\lambda)}{\sigma^2}} \Phi \left( e - \frac{2(\mu+\lambda)}{\sigma} \sqrt{u} \right)$$

with  $e$  defined as

$$e = \frac{-\ln(z) + \left( \mu + \lambda + \frac{\sigma^2}{2} \right) u}{\sigma\sqrt{u}}.$$

Using Mathematica, we have

$$\int_0^\infty e^{-(r+\lambda)u} g_{12}(z, u) du = -z^{-x_1}.$$

Hence, by Leibnitz' rule

$$\int_0^\infty e^{-(r+\lambda)u} g_2(z, u) du = \frac{1}{x_1 - 1} z^{1-x_1}.$$

The integration constant is zero. Because, for  $z = 1$ , Mathematica can evaluate  $\int_0^\infty e^{-(r+\lambda)u} g_2(1, u) du$  directly, and the result is  $\frac{1}{x_1 - 1}$ . We use these results to conclude that

$$\mathbb{E}_t \left[ \int_t^\infty e^{-(r+\lambda)(u-t)} dM_u \right] = \frac{X_t}{x_1 - 1} Z_t^{1-x_1}$$

and

$$\int_t^\infty e^{-(r+\lambda)(u-t)} \mathbb{E}_t[M_u] du = \frac{X_t}{r+\lambda} \left( Z_t + \frac{1}{x_1 - 1} Z_t^{1-x_1} \right).$$

Hence,

$$\mathcal{F}_{\text{nb}}(X_t, M_t) = \frac{aX_t}{r - \mu} - \frac{bX_t}{r + \lambda} Z_t - \frac{\left(\frac{b}{r + \lambda} - g\right)X_t}{x_1 - 1} Z_t^{1-x_1}.$$

The fundamental PDE (based on no arbitrage) for the general value function,  $\mathcal{F}(X, M)$ , in the dynamic case is very similar to the fundamental PDE for  $\mathcal{F}(X)$  in the static case. The reason is that the state variable  $X$  with probability one is strictly below  $M$ . Hence,  $M$  is a constant with probability one. However we need to specify proper boundary conditions when  $X = M$ . See the literature on Russian options. Hence, when  $X < M$ , the general value function,  $\mathcal{F}(X, M)$ , (with cash flow stream  $(aX_t + bM_t)dt + gdM_t$ ) must fulfill the fundamental PDE

$$\frac{1}{2}\sigma^2 X^2 \mathcal{F}_{11}(X, M) + (\mu + \lambda)X \mathcal{F}_1(X, M) - (r + \lambda)\mathcal{F}(X, M) + aX + bM = 0.$$

Moreover, using the homogeneity degree of one property of  $\mathcal{F}$ , we can simplify  $\mathcal{F}$  in the following way

$$\mathcal{F}(X, M) = X\mathcal{F}(1, Z) = XF(Z)$$

where

$$Z = \frac{M}{X}.$$

The (partial) derivatives of  $\mathcal{F}$  and  $F$  are related as follows,

$$\mathcal{F}_1(X, M) = F(Z) - ZF'(Z)$$

and

$$\mathcal{F}_{11}(X, M) = \frac{Z^2}{X} F''(Z).$$

Hence, when  $Z > 1$ , this function,  $F(Z)$ , must then fulfill the ODE

$$\frac{1}{2}\sigma^2 Z^2 F''(Z) - (\mu + \lambda)ZF'(Z) - (r - \mu)F(Z) + a + bZ = 0.$$

Hence, for  $Z > 1$ , the general solution is

$$F(Z) = f_1 Z^{1-x_1} + f_2 Z^{1-x_2} + \frac{a}{r - \mu} - \frac{bZ}{r + \lambda} - \frac{\frac{b}{r + \lambda} - g}{x_1 - 1} Z^{1-x_1}.$$

The boundary conditions for debt and equity are

$$\begin{aligned} E(\bar{Z}) &= 0, \\ E'(\bar{Z}) &= 0, \\ D(\bar{Z}) &= (1 - \alpha)A, \\ D(1) &= P, \\ E(1) &= A - P(1 - k), \end{aligned}$$

and

$$E'(1) = 0.$$

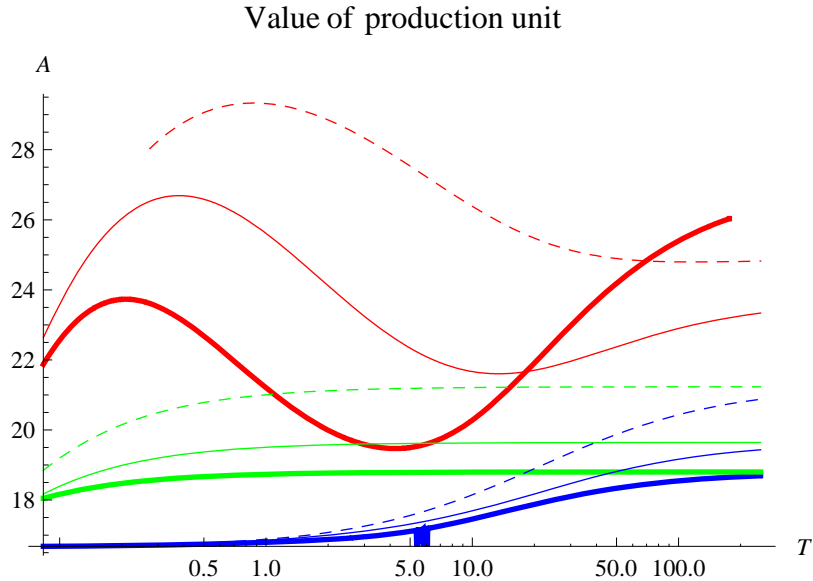


FIGURE 1. Red: Dynamic, Blue: Static, Green: Leland 98. Thick:  $\sigma = 50\%$ , Thin:  $\sigma = 25\%$ , Dashed:  $\sigma = 10\%$ .

##### 5. A GRAPHICAL RESULT AND SOME VERY PRELIMINARY EMPIRICAL RESULTS

In figure 1 we plot (in red) the value of the firm's newly issued debt and equity as a function of its chosen average maturity,  $\frac{1}{\lambda}$  of its debt for three different EBIT volatilities,  $\sigma$ . The figure shows that a firm with an EBIT volatility of 10% should choose an average maturity of around two years, whereas a firm with an EBIT volatility of 25% should choose an average maturity of around six months. Finally, a firm with an EBIT volatility of 50% should choose perpetual debt. For the firm with an EBIT volatility of 50% there is a local maximum of its value of (newly issued) debt and equity for an average maturity of 3 months, but it turns out that in this case the flexibility comes at a too high price, so the firm chooses to debt with no flexibility and then save on the issuing costs. The amount of debt issue issuing increases both as the maturity gets shorter and also as the volatility increases, so at some point, as the volatility increases, it suddenly will become optimal to choose perpetual debt for a given issuing cost,  $k$ .

In order to (at least in some sense) verify that this trade off is actually something that real firms take into account, we have regressed the duration of corporate debt in CRSP firms on the corresponding firms' earnings volatility, and indeed we find that the regression co-efficient is negative. That is, firms with higher EBIT volatility has debt with shorter average maturity.

##### REFERENCES