Risk, Uncertainty and Optimism in Venture Capital Relationships [*]

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Abstract

We develop a dynamic, structural model to address a long-standing puzzle, namely, the huge discrepancy between the discount rates used by venture capitalists to value projects ($\sim 50\%$) and the ex post realized returns of their investments ($\sim 15\%$). In our continuous-time, principal-agent framework, the venture capitalist (VC) and entrepreneur (EN) could have asymmetric beliefs about the intrinsic quality of a project in addition to asymmetric attitudes towards its risk. We indirectly infer the magnitude of EN optimism by matching statistics about the risks and returns of VC projects predicted by the model to their observed values. We demonstrate that EN optimism leads to "implied" discount rates for VCs precisely in the range reported in previous empirical research implying that the high discount rates used by VCs reflects an EN "optimism premium". Moreover, EN optimism substantially mitigates the agency costs of risk-sharing between the VC and EN and enhances the VC's expected payoff. Consequently, VCs have significant incentives to encourage EN optimism and these incentives also have a beneficial effect on the total value generated by the VC-EN relationship. Consistent with observed contractual structures, the equilibrium dynamic contract features both equity-like and debt-like components of the VC's payoff structure and the progressive vesting of the EN's stake in the firm. The duration, project value and the VC's expected payoff all increase with the project's transient risk but decrease with its intrinsic risk. Greater uncertainty about project quality enhances the value generated by VC-EN relationships.

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1 Introduction

A long-standing puzzle in the venture capital literature is the discrepancy between the high discount rates used by venture capitalists (VCs) and the significantly lower ex post realized returns of their investments (see Cochrane, 2005). Sahlman (1990) reports that VCs routinely use discount rates around 50% to value start-up projects, while Cochrane (2005) shows that the realized ex post returns of VC projects only average around 15%. Sahlman (1990) suggests that high discount rates could be a mechanism that VCs use to adjust for over-optimistic projections by entrepreneurs. Is entrepreneurial optimism in reality significant enough to generate *optimism premia* as high as 35% ? In this article, we develop a positive theory of venture capital investment which provides an affirmative answer to this unresolved question.

We build a dynamic, structural model that incorporates the possibility that VCs and entrepreneurs (ENs) could have asymmetric beliefs about the intrinsic quality of projects in addition to having asymmetric attitudes towards their risk. We estimate the key parameters of the model by matching statistics on the risk-return characteristics of projects predicted by the model to the data in Cochrane (2005). We indirectly infer the degree of EN optimism implied by the data and demonstrate that it leads to "implied" discount rates precisely in the range reported by Sahlman (1990). Our structural approach, therefore, shows that the risk-return characteristics of VC projects are, in fact, compatible with the high discount rates used by VCs in reality.

EN optimism could mitigate the agency costs of risk-sharing between VCs and EN's by 10-15% of firm value and enhance VC value by 15% suggesting that VCs have significant incentives to encourage or "feed" EN optimism, and these incentives also enhance firm value. The presence of EN optimism is a key determinant of the characteristics of VC relationships—their durations and economic values, the structures of dynamic contracts between VCs and ENs, and the staging of VC investment over time—and explains why discount rates used by VCs are much higher than those predicted by traditional asset pricing models.

In Section 3, we develop our continuous-time, principal-agent framework in which a cash-constrained, risk-averse EN with a project approaches a risk-neutral VC for financing. The VC manages a fund that has investments in several different projects. The capital managed by the fund is competitively supplied by outside investors who care about the expected excess return on assets of the fund over a benchmark in each period. We focus on a representative project in the VC fund's portfolio. The project generates value through physical capital investments by the VC and effort investments by the EN. The VC or the EN could terminate the relationship at any date.

All payoffs occur upon termination of the project.[1]

The state variable in the model is the project's *termination value* at each date, which is the total payoff if the project is terminated at that date. The termination value, which is contractible, evolves as a log-normal process. The drift of the termination value process has two components; a fixed component that represents the project's *intrinsic quality* and a varying, discretionary component that is determined by the VC's investments and the EN's effort. The VC and EN have imperfect information about the intrinsic quality of the project and could also differ in their beliefs about the quality with the EN being more optimistic. The beliefs are, however, *common knowledge*, that is, the VC and EN "agree to disagree". The VC's investments in the project are staged over time with future investment being contingent on intermediate observations of the project's termination value, which serve as "signals" that enable the VC and the EN to update their assessments of its quality in a Bayesian manner. The *degree of EN optimism* is the difference between the EN's and VC's mean assessments of project quality. The common variance of their assessments is the project's *transient* risk, which is resolved over time as the VC and EN update their assessments of the project's quality. The volatility of the termination value process is the project's *intrinsic risk* that remains invariant through time.

The EN has linear preferences with a stochastic discount rate that reflects his costs of bearing risk. He is provided with inter-temporal incentives to exert effort through a contract that specifies his share of the termination payoff of the project. Assuming the VC has the bargaining power, we derive the long-term, renegotiation-proof equilibrium contract between the VC and the EN, which describes the VC's investments over time, the EN's path-dependent payoff upon termination, and the inter-temporal performance targets that must be met for the relationship to continue. As the VC has the bargaining power, the relationship is terminated at the VC's behest in equilibrium.

We derive the equilibrium in Section 4. Under the equilibrium contract, the change in the EN's promised payoff over any period, which could also be viewed as his currently vested share in the project, is an affine function of the change in termination value or performance over the period. Conditional on continuation, the VC's investment as a proportion of the termination value or her proportional investment rate, the sensitivities of the EN's compensation to performance over each period (the pay-performance sensitivities), and the EN's effort in each period, are all deterministic functions of time. The fixed component of the change in the EN's promised payoff in each period is, however, stochastic and depends on the VC's and EN's current assessments of project quality. The optimal contract between the VC and EN predicted by the model shares many

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[1] Our analysis could be generalized to incorporate intermediate cash flows that are proportional to the termination values without altering our main results.

features with observed contractual structures: the VC's payoff structure has "debt" and "equity" components, the EN's stake in the project vests over time, and the VC's claim on the project's payoffs allows her to recover her cumulative investments with high probability (Sahlman, 1990, Kaplan and Stromberg, 2003).

We analytically derive properties of the equilibrium contract in Sections 5-7. The contract balances the costs of risk-sharing between the VC and the EN with the positive rents that the VC is able to extract due to the EN's optimism. Because the degree of EN optimism declines over time as the VC and EN learn about the project's intrinsic quality, the rents from the EN's optimism decrease relative to the costs of risk-sharing. The VC's proportional investment rates, the EN's pay-performance sensitivities, and the EN's effort, therefore, decrease monotonically over time. In the benchmark scenario in which the VC and EN have symmetric beliefs, however, the EN's pay-performance sensitivity and the VC's proportional investment rate are constant over time. Hence, the presence of asymmetric beliefs is a key determinant of the inter-temporal variations in the EN's pay-performance sensitivity and the VC's investment rate.

The paths of the VC's proportional investment rate, the EN's pay-performance sensitivity, and the EN's effort (conditional on continuation of the relationship) all decline with the project's intrinsic and transient risk and increase with the degree of EN optimism. The duration of the relationship increases with the degree of EN optimism and decreases with the EN's risk aversion. These results illustrate the negative effects of risk and the positive effects of EN optimism on risk sharing between the VC and EN and, therefore, the power of incentives for the EN. The negative relation between duration and the degree of EN optimism is consistent with the evidence in Kaplan and Stromberg (2003) that experienced entrepreneurs, who are likely to have more realistic beliefs, receive fewer rounds of financing.

In Section 8, we numerically implement the model to examine the central issues that motivate this study—how significant is EN optimism and could it generate the discrepancy between observed VC discount rates and the average ex post returns of VC projects ? We determine the baseline values of the key structural parameters of the model, which include the degree of EN optimism, his cost of bearing risk and his disutility of effort. We indirectly infer the values of these parameters by matching the average risks and returns of projects in each period or "round" predicted by the model to their corresponding observed values reported in Cochrane (2005). The model is able to match the risk-return characteristics of projects documented by Cochrane (2005) quite well. The baseline model also matches the data on the distribution of value generated by VC investments reported in Sahlman (1990), which demonstrates the model's out-of-sample predictive ability.

Our indirect inference approach shows that the average initial (that is, at the inception of a project) degree of EN optimism implied by the data is approximately 33.5%. We define the *implied discount rate* for a project as the rate at which the EN's projections about the payoffs from a project would have to be discounted so that the resulting project value is equal to the project value under the VC's projections of the project's payoffs. Consistent with the average expected return of VC projects documented in Cochrane (2005), we assume a baseline discount rate or "cost of capital" of 15% to obtain the project value under the VC's projections of its payoffs. We vary parameters describing project characteristics about their baseline values and calculate the corresponding implied discount rates allowing for the cost of capital to vary with the project's intrinsic risk.

The implied discount rates for a wide range of projects cluster around 50%, which is quite consistent with the ranges of VC discount rates reported in Sahlman (1990). Our study, therefore, confirms that entrepreneurial *optimism premia* are high enough to justify the discount rates used by VCs. The discrepancy between the high discount rates and the low realized expected returns is due to the fact that the discount rates are used to discount *entrepreneurs' projections* of the payoffs of projects rather than VCs' more realistic projections. If VC's valued projects using their own projections, the corresponding discount rates are much lower and consistent with the expected returns of VC projects reported in Cochrane (2005). The stability of the predicted implied discount rates suggests that a discount rate of 50% is a remarkably good "rule of thumb" to value VC projects.

In Section 9, we numerically derive several additional implications of the model by examining the effects of varying parameters about their baseline values on the duration of the relationship, firm value, and the net present value of the payoffs to the VC fund or the *VC value*. We compare our results to the "no agency" benchmark scenario in which there are no conflicts of interest between the VC and EN as well as the "symmetric beliefs" benchmark scenario in which the VC and EN have asymmetric attitudes towards risk, but have symmetric beliefs about project quality.

Consistent with our earlier analytical results, EN optimism significantly increases firm value, the VC value, and the duration of the relationship. In the baseline case, EN optimism reduces the agency costs of risk sharing by 12% and increases VC value by 15%. The increase in the VC value due to EN optimism is generally greater than the increase in firm value, which reflects the substantial rents that the VC extracts by "feeding" EN optimism. The positive effects of EN optimism on firm value are consistent with the empirical evidence reported in Gelderen, Thurik and Bosma (2005). We also find that firm value is positively related to the duration of the relationship, which is consistent with the evidence in Gompers (1995).

The expected duration, firm value and VC value all increase with the project's human capital intensity because the EN's effort is a key driver of the economic value generated by the VC relationship. The project's intrinsic and transient risks have opposing effects on the "speed of learning" about project quality and, therefore, the rate at which the degree of EN optimism declines over time. As a result, they have dramatically *opposite* effects on the project's duration, firm value, and the VC value. All three variables increase with the project's initial transient risk, but decrease with its intrinsic risk. Hence, firm value and the VC value are enhanced when there is greater uncertainty about project quality. Consistent with empirical evidence, therefore, VCs have significant incentives to finance innovative projects compared with mature or "imitating" projects because there is greater uncertainty about the quality of innovative projects (Hellman and Puri, 2000).

In summary, our study shows that entrepreneurial optimism is an important determinant of the characteristics of VC relationships. The tractability of our structural model, as well as the fact that it is able to match empirical data on the risk-return characteristics of projects and the discount rates used to value them, suggests that it could be useful as a tool to value risky ventures whose cash flows are affected by asymmetric beliefs and agency conflicts. In this respect, our study belongs to the growing literature at the interface between asset pricing theory and corporate finance that values claims to risky cash flows by explicitly modeling the actions of the agents who generate and share cash flows.

2 Related Literature

While the effects of agency conflicts and imperfect information are studied in several contexts by previous studies, theoretical literature that incorporates asymmetric beliefs in the context of venture capital investment is relatively nascent. Landier and Thesmar (2005) develop a two-period VC model with asymmetric beliefs. They show that optimistic entrepreneurs tend to rely on short term debt rather than long term debt. Our framework differs significantly from theirs in that investment could be staged over time and the dynamic contract between the VC and the EN has both "equity" and "debt" components. Cuny and Talmor (2005) also analyze the effects of asymmetric beliefs in a two-period VC finance model that compares the performance of firms funded by milestone staging to those funded by investment rounds. They find that the advantages associated with round financing are increased when the EN is more optimistic than the VC. The focus of their study is on analyzing the effect of asymmetric beliefs on the two types of financing mentioned above. We examine how the interplay between asymmetric beliefs and agency conflicts affect the manner in which VC investment is staged over time, the dynamic contract between the VC and the EN, and the duration and economic value of the VC-EN relationship.

A second stream of the literature investigates the importance of staging in mitigating VC-EN agency conflicts. In a deterministic model, Neher (1999) shows that staging is essential to overcome the hold-up problem. As in Neher (1999), the manner in which VC investment is staged over time as well as the number of stages are determined endogenously in our framework. As Neher's (1999) model is fully deterministic, however, his framework cannot be used to study the effects of risk, imperfect information, and asymmetric beliefs on the characteristics of VC-EN relationships.[2]

A third strand of the literature analyzes the features of the optimal contracts that emerge in "double sided" two-period moral hazard models in which the VC and EN exert effort (Casamatta, 2003, Cornelli and Yosha, 2003, Repullo and Suarez, 2004, Inderst and Muller, 2004, Renucci, 2005). We too develop a model in which the VC and EN take value-enhancing actions. Similar to these studies, the optimal contracts predicted by our analysis have "debt" and "equity" features consistent with observed contractual structures. In sharp contrast, however, our analysis focuses on the effects of asymmetric beliefs on the characteristics of VC-EN relationships. [3]

There is a vast and rapidly growing literature that develops and analyzes dynamic principal-agent models in various economic contexts (see Laffont and Martimort, 2002 and Ljungqvist and Sargent, 2004 for surveys). Our framework shares features of models examined in this literature that incorporate *external* imperfect information and Bayesian learning (for example, Gibbons and Murphy, 1992, Bergemann and Hege, 1998, Holmstrom, 1999). We contribute to this literature by developing and analyzing a framework where both the VC (the principal) and the EN (the agent) make investments (physical and human capital) over time, have asymmetric beliefs about project quality, and the relationship is terminated endogenously. The distinction between projects' intrinsic and transient risks that we emphasize in our analysis leads to novel implications for the effects of these two components of risk on VC relationships.

We differ significantly from studies in the above-mentioned streams of the literature in that we focus on *quantifying* the effects of entrepreneurial optimism implied by the data on the risk-return characteristics of VC projects. The analysis of our structural model explains the huge discrepancy between VC discount rates and the average ex post returns of VC projects. The predictions of our study are broadly consistent with extant empirical evidence on the characteristics of VC-EN relationships and also suggest additional testable implications that could provide guidance for future empirical research.

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[2] Kockesen and Ozerturk (2004) argue that some sort of EN "lock-in" is essential for staged financing to occur. Egli, Ongena and Smith (2005) argue that staging can be used to build an EN's credit rating. Berk, Green and Naik (2003) develop an R&D model with a single, monolithic agent in which staging is exogenous

[3] Admati and Pfleiderer (1994) analyze a two-period model of venture capital investment and show that a fixed fraction contract is optimal. Fluck, Garrison and Myers (2005) show that a fixed-fraction contract may not be optimal if the EN's effort is endogenously determined. Trester (1998) shows that the presence of asymmetric information leads to the optimality of preferred equity over debt contracts.

3 The Model

We consider a continuous time framework with time horizon $[0, T]$. Apart from its analytical tractability, a continuous time formulation circumvents the delicate issues related to the existence of optimal contracts that arise in discrete-time frameworks in which payoffs are drawn from continuous distributions with unbounded supports (see, for example, Holmstrom and Milgrom, 1987). At date 0, a cash-constrained EN with a project approaches a VC for funding. The VC is the manager of a venture capital fund with a diversified portfolio of projects. The fund's capital is provided by diffuse outside investors. We focus on a representative project in the VC fund's portfolio. The project can potentially generate value through physical capital investments by the VC and human capital (effort) investments by the EN. The VC and the EN have imperfect information about the project and differ in their initial assessments of the project's quality.

If the VC agrees to invest in the project, she invests an initial amount of "seed" capital V_0 in the project and offers the EN a long-term contract that describes her subsequent investments in the project over time and the EN's compensation. To simplify the analysis and notation, we assume that the VC's investments are made continuously over time. (All our results extend to the scenario where investments are made at discrete intervals.)

Either the VC or the EN could terminate the relationship at any date. The key state variable in the model is the total payoff V_t if the VC-EN relationship is terminated at date t.[4] For simplicity, we assume the project does not generate any intermediate cash flows. All our results hold in an extension of the model in which the project generates a cash flow $\delta V_t dt$ in period $[t, t + dt]$ where $\delta > 0$ is a constant. Hereafter, we refer to the variable V_t as the project's *termination value* at date t. The termination value at any date is observable and verifiable and, therefore, contractible.[5]

The *incremental termination value*, that is, the change in termination value over any period $[t, t + dt]$, depends on the investment by the VC, the amount of effort exerted by the EN, the intrinsic quality of the project, and its risk. The VC closely monitors the EN so that the EN's effort is observable to the VC.[6] However, it is non-verifiable by a third party and, therefore, not

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[4] The termination value is the amount outside investors would be willing to pay at date t to obtain future earnings from the project. The VC and the EN possess project-specific skills that are not transferrable. Neither the VC nor the EN can commit to supplying these skills to a third party. Hence, the amount outside investors would be willing to pay for the project is, in general, lower than the value if full commitment by the VC and the EN were hypothetically possible. The value under full commitment is the *rational expectations* value of the project, which rationally incorporates the effects of future physical capital investments by the VC and human capital investments by the EN as well as the endogenous decision to terminate the project.

[5] An important stream of the literature examines the effect of "incomplete contracting" on VC relationships (Aghion and Bolton, 1992, Kirilenko, 2001).

[6] Sahlman (1990) reports that venture capitalists closely monitor the firms they invest in. He mentions that lead venture investors visit each company in their portfolio 19 times every year and spend 100 hours in direct contact. Hellman and Puri (2000, 2002) emphasize the importance of monitoring and oversight in their empirical analyses of venture capital financing.

directly contractible. Since the termination value is the only economic variable that is contractible, the EN must be provided with appropriate incentives to exert effort through a contract that depends on the termination value process. Both the VC and the EN have imperfect information about the intrinsic quality of the project, but have priors on it that differ from each other in general. The VC and the EN update their assessments of the project's intrinsic quality in a Bayesian manner based on their observations of the termination values, VC's investments, and the EN's effort.

We begin by first describing how the VC's physical capital investments and the EN's human capital (effort) investments affect the project's termination value over time.

3.1 The Termination Value Process

The initial termination value of the project equals the initial seed capital investment V_0 of the VC. Suppose the VC invests capital $c_t V_t dt$ and the EN exerts effort $\eta_t V_t dt$ over the time interval $[t, t+dt]$ so that c_t and η_t are the investment and effort rates *as proportions* of the state variable V_t. Hereafter, we refer to c_t and η_t as simply the investment rate and effort, respectively, to simplify the exposition.

If the EN's effort and the VC's investment are both positive, the project's termination value evolves as:

$$dV_t = (c_t^{\alpha} \eta_t^{\beta} - l_t) V_t dt \ + \ \Theta V_t dt \ + \ s V_t dB_t. \tag{1}$$

In the above, B is a standard Brownian motion. If either the EN's effort or the VC's investment is zero, then the change in termination value is given by $dV_t = -l_t V_t dt$. It is not difficult to show that, in equilibrium, effort and investment are both positive if the project continues. We simplify the notation and exposition by henceforth assuming the termination value evolves as in (1).

The change in termination value is derived from three sources: "net discretionary output", "project quality" and "intrinsic risk".

Net discretionary output: Discretionary output in period $[t, t+dt]$ is a direct result of the VC's investment rate $c_t V_t$ and the EN's effort η_t, and is described by the function $c_t^{\alpha} \eta_t^{\beta}$, $\alpha, \beta > 0$. Net discretionary output is discretionary output less the "operating costs", which are represented by the term $l_t V_t dt$. The operating costs could include wages to salaried employees, depreciation expenses, decline in revenues due to increased competition, fixed costs arising from increases in the scale of the project, etcetera. The operating costs parameter l_t is deterministic, increasing and convex over time, which ensures that termination occurs in finite time almost surely.

Intrinsic risk: The term $s V_t dB_t$, where $s > 0$ is constant, represents the "intrinsic" component of the project's risk in period $[t, t+dt]$. The intrinsic risk is the component of the project's risk that remains invariant over time.

Project quality: The parameter Θ represents the growth rate of the project's termination value arising from the project's intrinsic quality. The VC and the EN have imperfect information about Θ and could also differ in their beliefs about its value. Their respective beliefs are, however, *common knowledge*; that is, they "agree to disagree". The uncertainty in the value of Θ may be viewed as the project's *transient risk*. The transient risk is resolved over time as the VC and the EN update their priors on Θ in a Bayesian manner based on observations of the project's performance.

The VC's and EN's initial priors on Θ are normally distributed with $\Theta \sim N(\mu_0^{VC}, \sigma_0^2)$ and $\Theta \sim N(\mu_0^{EN}, \sigma_0^2)$, respectively. Define

$$\xi_t dt := d\ln V_t - (c_t^\alpha \eta_t^\beta - 0.5s^2 - l_t)dt = \Theta dt + s dB_t. \tag{2}$$

Since the VC's investment rate c_t and the EN's effort η_t are observable, it follows from well-known formulae (DeGroot 1970, Oksendal 2003) that the posterior distribution on Θ for each date $t \geq 0$ is $N(\mu_t^\ell, \sigma_t^2)$, $\ell = VC, EN$, where

$$\sigma_t^2 = \frac{s^2\sigma_0^2}{s^2 + t\sigma_0^2} \,, \tag{3}$$

$$\mu_t^\ell = \frac{s^2\mu_0^\ell + \sigma_0^2 \int_{u=0}^t \xi_u du}{s^2 + t\sigma_0^2} \,,\ \ell = VC, EN. \tag{4}$$

From Oksendal (2003), we can show that the evolution of the mean posterior assessment of project quality, μ_t^ℓ, is described by the following stochastic differential equation

$$d\mu_t^\ell = \frac{\sigma_0^2}{s^2 + t\sigma_0^2}[d\ln V_t - (c_t^\alpha \eta_t^\beta - 0.5s^2 - l_t)dt - \mu_t^\ell dt] \,,\ \ell = VC, EN. \tag{5}$$

From (1) and (5), we see that the standard deviation σ_t^μ of the evolution of the mean assessment of project quality is

$$\sigma_t^\mu = \frac{s\sigma_0^2}{s^2 + t\sigma_0^2}. \tag{6}$$

Note that the standard deviations of the evolutions of the VC's and EN's mean assessments of project quality are equal and decrease with time. Let

$$\Delta_t := \mu_t^{EN} - \mu_t^{VC} = \frac{s^2\Delta_0}{s^2 + t\sigma_0^2} = \frac{\sigma_t^2}{\sigma_0^2}\,\Delta_0 \tag{7}$$

denote the *degree of asymmetry in beliefs* at date t. It follows from (7) that the degree of asymmetry in beliefs is resolved deterministically over time, and there is a linear relationship between the resolution of the asymmetry of beliefs and the resolution of the transient risk. Consistent with the evidence reported by Landier and Thesmar (2005), Sahlman (1990) and other researchers, we assume that the EN is initially more confident of the success of his ideas so that $\Delta_0 \geq 0$. Thus, Δ_t decreases with time since the variance σ_t^2 decreases with time.

For future reference, we denote the information contained in the history of termination values $\{V_t,\ t \geq 0\}$ by $\{\mathcal{F}_t\}$. We let $\{\mathcal{G}_t\}$ denote the information contained in the history of termination values, effort choices by the EN, and capital investments by the VC, which is known to both the VC and EN. Clearly, $\mathcal{F}_t \subset \mathcal{G}_t$.

3.2 Contracting between the VC and EN

Since the project does not generate intermediate cash-flows, the contract between the VC and the EN describes the VC's incremental capital investments over time and the payoffs to be received by both parties upon termination. Further, since either the VC or the EN could choose to terminate the relationship at any date, the contract specifies the payoffs to be received by both parties if the project were terminated at any date $t \geq 0$. More precisely, a feasible contract is described by the triplet (P, c, τ), where P_t is the EN's payoff and $V_t - P_t$ is the payoff to the VC fund if the relationship is terminated at date $t \geq 0$, c_t is the VC's investment rate at date t, and τ is the termination time. Note that $V_t - P_t$ is the payoff to the VC *fund* upon termination, and *not* the VC's compensation, which we describe shortly. Since the initial termination value V_0 of the project equals the VC's initial capital investment, $P_0 = 0$. As the project's termination value is the only economic quantity that is contractible, the processes P and c are $\{\mathcal{F}_t\}$-adapted, and τ is an $\{\mathcal{F}_t\}$ stopping time.

The VC, who possesses the bargaining power, offers the EN a long-term contract at date zero, which specifies her dynamic capital investments in the projects, the termination time, and the EN's payoff upon termination. The EN, in turn, dynamically chooses his effort to maximize her total expected utility including her disutility from effort.

3.3 The VC's Objective

The VC is the manager of a venture capital fund with capital supplied by outside investors. The fund has investments in several projects. The investors in the venture capital fund care about the performance of the fund relative to a benchmark, specifically, the return on the fund's assets in excess of the benchmark return. As we focus on a representative project in the VC fund's portfolio, we define the relative performance in terms of the performance of the representative project.[7]

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[7] In this paper, we focus on a representative VC and project and are not directly concerned with heterogeneity across VCs. Sorensen (2006) estimates a structural two-sided matching model and shows that more experienced VCs chooses better projects. Similar to our study, Dessi (2005) also analyzes a 3-tier model in which the investors in a VC fund, the VC and EN are all distinct economic agents.

The project's "outside" value or asset value is V_t at any date t. The value of the fund's holdings in the project at date t is the termination value less the promised payoff to the EN, $V_t - P_t$. The change in the value of the fund's holdings over the period $[t, t+dt]$ (if the project is continued), after subtracting the capital investment $c_t V_t dt$ specified by the contract, is $[V_{t+dt} - P_{t+dt} - c_t V_t dt] - [V_t - P_t]$. The change in the value of the fund's holdings from instead investing in the benchmark is $V_t R_b dt$ where R_b is the benchmark return on assets. The benchmark return R_b could be viewed as the "cost of capital" for the project. The return on assets of the fund in excess of the benchmark over the period is

$$R_{t+dt} - R_b dt = \frac{[V_{t+dt} - P_{t+dt} - c_t V_t dt] - [V_t - P_t]}{V_t} - \frac{V_t R_b dt}{V_t} = \frac{dV_t - c_t V_t dt - dP_t}{V_t} - R_b dt. \quad (8)$$

In Appendix A, we model the relationship between outside investors and the VC and show that the endogenously derived assets under management at date t are an affine function of the expected excess return on assets.[8] We assume the VC receives a constant proportion of the assets under management as compensation so that her payoff over the period is also an affine function $A + BE_t(R_{t+dt} - R_b dt)$, $A \geq 0$, $B > 0$, of the expected excess return on assets.[9] Since the fixed portion of the VC's compensation neither influences nor is affected by her actions, we hereafter set $A = 0$ to simplify the notation.

The VC is risk-neutral so that her discounted expected utility from investing in the project until termination is

$$\text{VC's Expected Utility } = E_0^{VC}\Big[\int_0^\tau e^{-rt} B E_t(R_{t+dt} - R_b dt)\Big] = E\Big[\int_0^\tau e^{-rt} B(R_{t+dt} - R_b dt)\Big], \quad (9)$$

where the expectation is under the VC's beliefs about the project's intrinsic quality, and the second equality above follows from the law of iterated expectations. The risk-free rate r is a constant. Because B is merely a constant of proportionality, we henceforth set it equal to one.

3.4 The EN's Objective

As in several studies in the principal-agent literature, the EN has linear inter-temporal preferences with a subjective discount rate (for example, DeMarzo and Fishman, 2003). In contrast with these studies, however, the EN's subjective discount rate is *stochastic*, which directly reflects his costs of bearing risk. The EN incurs an additive disutility of effort, which is expressed in monetary terms.

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[8] Our results also hold if the assets under management are any increasing, deterministic function of the expected excess return.

[9] VC managers typically receive a proportion of total committed capital as well as a percentage of profits. The constant "proportion of assets" compensation structure we assume here is analogous to what is used by mutual fund companies and greatly facilitates the model's tractability.

More precisely, if the EN's termination payoff is P_τ and his effort is described by the process $\{\eta_t\}$, his subjective valuation of his future payoffs (including his disutility of effort) at date zero is given by

$$U_0 = E_0^{EN}\Big[e^{-r\tau-\frac{1}{2}\lambda^2\tau-\lambda B_\tau}P_\tau - \int_0^\tau e^{-rt-\frac{1}{2}\lambda^2 t-\lambda B_t}k\eta_t^\gamma V_t dt\Big]. \tag{10}$$

In (10), $e^{-rt-\frac{1}{2}\lambda^2 t-\lambda B_t}$ is the stochastic discount factor by which the EN discounts a payoff at date t, where $\lambda > 0$ measures his cost of bearing risk. In Section 8, we estimate λ by matching the model's predictions to data. The expectation in (10) is with respect to the EN's beliefs about the project's intrinsic quality. The term $k\eta_t^\gamma V_t dt; k > 0$ represents the EN's disutility of effort in period $[t, t+dt]$. The EN's disutility of effort increases with the "scale" of the project represented by its termination value. This is consistent with the fact that that the expected change in termination value due to investment and effort is also proportional to the termination value—see (1). As in the case of the termination payoff, the EN values his cost of effort in any period by its expectation weighted by the stochastic discount factor.

In our subsequent analysis, we find it useful to express the EN's subjective valuation (10) of his future payaoffs in an alternate form. The process $\exp(-\frac{1}{2}\lambda^2 t - \lambda B_t)$ is a square-integrable martingale, and is the Radon-Nikodym derivative process of a new probability measure equivalent to the original one (see Chapter 6 of Duffie, 2001). The EN's valuation (10) can then be expressed as

$$U_0 = \overline{E}_0^{EN}\Big[e^{-r\tau}P_\tau - \int_0^\tau e^{-rt}k\eta_t^\gamma V_t dt\Big], \tag{11}$$

where the expectation above is under the new probability measure. From (11), we see that we can express the EN's objective by assuming that he is risk-neutral, *not under the actual probability*, but under a risk-adjusted probability, the *EN's valuation probability*, which reflects his costs of bearing risk.

For future reference, we note that, by Girsanov's theorem (see Chapter 6 of Duffie, 2001), the termination value process evolves as follows under the EN's valuation probability:

$$dV_t^{EN} = (c_t^\alpha \eta_t^\beta - l_t - \lambda s)V_t dt \; + \; \Theta V_t dt \; + \; sV_t dB_t^{EN}. \tag{12}$$

In (12),

$$B_t^{EN} = B_t + \lambda t \tag{13}$$

is a Brownian motion under the EN's valuation probability. The superscripts on this process and the termination value process indicate that the evolution (12) is in the EN's valuation probability.

In Appendix A, we show that the EN's preferences as described by (10) or (11) actually belong to the general class of recursive or "stochastic differential" preferences analyzed by Duffie and Epstein (1992).

3.5 The VC-EN Interaction

At each date t, the EN could terminate the relationship with the VC and receive his promised payoff P_t. The EN chooses to continue the relationship over the next infinitesimal time period if and only if his expected utility from continuation exceeds his utility from termination. The *continuation value* of the EN at date t, CU_t, is

$$CU_t := \overline{E}_t^{EN}\left[e^{-r(\tau-t)}P_\tau - P_t - \int_t^\tau e^{-r(u-t)}k\eta_u^\gamma V_u du\right] \tag{14}$$

where $\overline{E}_t^{EN}$ denotes the EN's expectation under his valuation probability conditioned on the information available at date t.

Similarly, the risk-neutral VC chooses to continue the relationship if and only if her expected stream of future payoffs from continuing the relationship exceeds her payoff from termination. From (9) (recall that we have set $B = 1$), the VC's continuation value CV_t at date t is given by

$$CV_t := E_t^{VC}\left[\int_t^\tau e^{-r(u-t)}\big(\frac{dV_u - dP_u - c_u V_u du}{V_u} - R_b\big)du\right], \tag{15}$$

where E_t^{VC} denotes the expectation under the VC's beliefs conditioned on the information available at date t. The VC chooses to continue the relationship at date t if and only if her continuation value is non-negative.

4 Equilibrium

The following conditions on the parameters will be assumed for the remainder of the paper:

Assumption 1 $(1-\alpha)\gamma/\beta > 2$.

Assumption 2 $\Delta_0 < \lambda s$.

These conditions ensure that an equilibrium contract between the VC and the EN exists and is unique and stable. The first assumption ensures that the curvature of the EN's disutility of effort is above a threshold relative to the sensitivity of output to his effort. Assumption 2 ensures that EN optimism is not high enough to outweigh the costs of risk-sharing.

4.1 Functional Form of Optimal Contract

In Appendix B (see Lemma 1), we show that the optimal contract has the following affine form:

$$dP_t = a_t V_t dt + b_t dV_t. \tag{16}$$

In (16), the contractual parameters $a_t \in R$, $b_t \in R_{++}$ are $\{\mathcal{F}_t\}$-measurable. It follows easily from (16) that the EN's contract P satisfies

$$P_\tau = P_0 + \int_{t=0}^{\tau} [a_t V_t dt + b_t dV_t]. \tag{17}$$

In Lemma 1 in Appendix B, we also show that the optimal long-term contract between the VC and the EN can be implemented by a sequence of single-period contracts, which are negotiated at each date t with the VC making "take it or leave it" offers to the EN.

4.2 Existence and Characterization of Equilibrium

We now turn to deriving the equilibrium contract, which is summarized in Theorem 1. We deliberately sacrifice some mathematical rigor in our analysis to simplify the exposition. We provide the detailed mathematical arguments underlying the derivation of the equilibrium in Appendix B.

We use backward induction to characterize the equilibrium by first considering the last possible (infinitesimal) investment period $[T-dt, T]$. Suppose that the project has not been terminated as of the date $T-dt$. Recall that the EN and VC priors on Θ as of date $T-dt$ are $N(\mu^{\ell}_{T-dt}, \sigma^2_{T-dt})$ with μ^{ℓ}_{T-dt} and σ^2_{T-dt} given by (4) and (3), respectively with the index t set to $T-dt$. Keep in mind that the VC's and EN's initial beliefs and, therefore, their respective beliefs at any date are common knowledge. For subsequent convenience in our inductive derivation of the equilibrium, it will be convenient to use the index t to denote the date. The index $t = T - dt$ for now, but it will later denote an arbitrary date when we establish the inductive step in our analysis.

The EN's Optimal Effort in Period $[T-dt, T]$ for a Given Contract

Suppose that in period $[t, t+dt]$ (recall that $t = T - dt$), the VC's investment rate is c and the EN's contractual parameters are (a, b)—see (16). If the EN's effort is η in period $[t, t+dt]$, his continuation value (14) is given by

$$CU_t = \overline{E}^{EN}_t (aV_t dt + bdV_t - k\eta^\gamma V_t dt). \tag{18}$$

From (12), we have

$$CU_t = (a + b(c^\alpha \eta^\beta - l_t + \mu^{EN}_t - \lambda s) - k\eta^\gamma) V_t dt\ . \tag{19}$$

By Assumption 1, the optimal effort level exists and is given by

$$\eta(b, c) := \Big(\frac{\beta c^\alpha b}{\gamma k}\Big)^{\frac{1}{\gamma-\beta}}. \tag{20}$$

The VC's Choice of Contract in Period $[T-dt,T]$

The VC will choose her investment rate c and the EN's contractual parameters (a,b) that will ensure the EN's participation and also rationally anticipate the EN's best effort response. Since the VC has the bargaining power and can make a "take it or leave it" offer to the EN, it is optimal for her to choose (a,b) so that regardless of the state at date $t=T-dt$

$$CU_t \equiv 0. \tag{21}$$

It follows from (19) that the relation between the parameters a, b, and c in period $[t,t+dt]$ that ensures (21) is given by

$$a(b,c) := k\eta(b,c)^{\gamma} - b(c^{\alpha}\eta(b,c)^{\beta} - l_t - \lambda s + \mu_t^{EN}). \tag{22}$$

Incorporating the EN's best effort response, the VC's continuation value (15) at date t is

$$CV_t = E_t^{VC}\big[\frac{(1-b)dV_t - a(b,c)V_t dt - cV_t dt}{V_t} - R_b dt\big]. \tag{23}$$

Substituting the EN's optimal effort (20) into (23),

$$CV_t = \Lambda_t(b,c)dt \tag{24}$$

where

$$\Lambda_t(b,c) := (\Delta_t b - \lambda s b + \phi(b)c^{\alpha\frac{\gamma}{\gamma-\beta}} - c \;+\; \mu_t^{VC} - l_t - R_b)dt \tag{25}$$

In (25), Δ_t is the degree of asymmetry in beliefs at date t, defined in (7), and

$$\phi(b) \;:=\; \left(\frac{1}{k}\right)^{\frac{\beta}{\gamma-\beta}}\left(\left(\frac{\beta b}{\gamma}\right)^{\frac{\beta}{\gamma-\beta}}\left(1-\frac{\beta b}{\gamma}\right)\right). \tag{26}$$

The VC chooses the capital investment rate c and the EN's pay performance sensitivity b to maximize $\Lambda_t(b,c)$. We first determine the VC's optimal investment rate $c(b)$ as a function of the EN's pay-performance sensitivity b. We then derive the optimal pay-performance sensitivity and the corresponding investment rate.

The function $\phi(b)$ in (26) is nonpositive when $b \geq \gamma/\beta$. Given that Δ_t decreases with t, it follows directly from Assumption 2 and (25) that the VC will never choose a pay performance sensitivity $b \geq \gamma/\beta$. For $b \in (0,\gamma/\beta)$, $\phi(b) > 0$ and Assumption 1 guarantees that $\Lambda_t(b,\cdot)$ is strictly concave in c since the exponent on c is guaranteed to be less than 1. Consequently,

$$c(b) \;=\; \hat{K}\phi(b)^{\frac{\gamma-\beta}{(1-\alpha)\gamma-\beta}} \tag{27}$$

$$\Lambda_t(b,c(b)) \;=\; \Delta_t b - \lambda s b + Kc(b) + (\mu_t^{VC} - l_t - R_b)\,. \tag{28}$$

The constants $\hat{K}$ and K in (27) and (28) are positive and depend on α, β and γ. We conclude that the VC will choose the pay performance sensitivity in period $[t, t+dt]$ to solve

$$b_t^* := \arg \max_{0<b<\gamma/\beta} \Lambda_t(b, c(b)). \tag{29}$$

In addition, we have the following characterization.

Proposition 1

(a) Under Assumption (1) the optimal investment function $c(\cdot)$ *is positive, increasing and strictly concave on* $(0,1]$*, is decreasing on* $[1, \gamma/\beta]$ *and therefore achieves its maximum at* $b = 1$*.*

(b) Under Assumptions (1) and (2) there is a unique solution $b_t^* \in (0,1)$ *to (29).*

The Inductive Step

We now set $t = T - 2dt$, and suppose the project has not been terminated as of date $(T - 2dt)$. If the VC's investment rate is c, the EN's contractual parameters are (a, b), and he exerts effort η, his continuation value (14) is given by

$$\begin{aligned} CU_t &= \overline{E}_t^{EN}\left[(aV_t dt + bdV_t - k\eta^\gamma V_t dt) + e^{-rdt} CU_{t+dt}\right] \\ &= \overline{E}_t^{EN}\left[(aV_t dt + bdV_t - k\eta^\gamma V_t dt)\right] . \end{aligned} \tag{30}$$

The first line above follows by the law of iterated expectations and the second line follows from (21). Since (30) is identical to (18), our previous arguments show that the EN's optimal effort $\eta(b, c)$ is given by (20) and the component $a(b, c)$ of the EN's compensation is given by (22).

It remains to determine the VC's optimal choices for the investment rate and pay performance sensitivity. Incorporating the EN's best effort response, the VC's continuation value at date t is (see 15)

$$CV_t = E_t^{VC}\left[\frac{(1-b)dV_t - a(b,c)V_t dt - cV_t dt - R_b V_t dt}{V_t} + e^{-rdt}\max(CV_{t+dt}, 0)\right] \tag{31}$$

$$= \Lambda_t(b, c)dt + e^{-rdt} E_t^{VC} \max(CV_{t+dt}, 0) , \tag{32}$$

where the second line follows from (23) and (24). Since the EN's effort is observable, the updated assessments of project quality at date $t + dt$, μ_{t+dt}^{EN}, do not depend on the VC's investment or the EN's effort over the period $t + dt$. It then follows from (24) and (25) that the second term on the right hand side of (32) does not depend on the EN's effort or the VC's investment at date t. Hence, the VC's continuation value at date t is maximized when the optimal investment is given by (27) and the optimal pay performance sensitivity solves (29).

We can clearly extend the above arguments by induction to any date t and thereby derive the equilibrium, as characterized in the following theorem.

Theorem 1 (Characterization of Equilibrium)

Under Assumption 1, if the project has not been terminated as of date $t \in [0,T]$, then an equilibrium contract offered by the VC and the EN's effort in the period is characterized, as follows:

- *The pay performance sensitivity is b_t^*, the solution to (29);*
- *The investment rate is $c_t^* := c(b_t^*)$ where $c(\cdot)$ is defined in (27);*
- *The fixed portion of the EN's compensation is $a_t^* := a(b_t^*, c_t^*)$ defined in (22);*
- *The optimal effort level is $\eta_t^* := \eta(b_t^*, c_t^*)$ defined in (20).*
- *The VC's maximum continuation value at date t is given by*

$$CV_t = \overbrace{\Lambda_t(b_t^*, c_t^*)dt}^{\textit{within-period flow}} + e^{-rdt}\overbrace{E_t^{VC}\left[\max\{CV_{t+dt}, 0\}\right]}^{\textit{future option value}}. \tag{33}$$

4.3 The VC's Objective Function

Since the degree of asymmetry in beliefs, Δ_t, and variance, σ_t^2, are deterministic functions of time—see (7)—it follows from Proposition (1) and Theorem 1 that the equilibrium values for the pay performance sensitivity, investment and effort at each point in time (conditional upon continuation) are positive and also deterministic. The only component of the contract that is stochastic and is adjusted based on realizations of the termination value V_t of the project (the "signal" of project quality) is the *fixed* component a_t^* of the EN's compensation. Furthermore, the equilibrium described in Theorem 1 is *stable*—the EN's pay performance sensitivities, effort, and the VC's investment rates are continuous functions of the model parameters.

As summarized in Theorem 1, the equilibrium contract at date t is determined by b_t^*. Let

$$F_t(b) := \Lambda_t(b, c(b)) - \mu_t^{VC} + l_t + R_b = \Delta_t b - pb + Kc(b) \tag{34}$$

denote the "controllable" portion of the "within-period flow" in period $[t, t+dt]$, and define

$$F_t^* := \max_{0<b<1} F_t(b). \tag{35}$$

Clearly, the solution to (29) is also a solution to (35). The *VC's objective function*, $F_t(b)$, consists of three components:

Economic rent from the EN's optimism. The term, $\Delta_t b$, reflects the rents that the VC extracts from the EN by exploiting his optimism about the project's intrinsic quality.

Cost of risk. The term, λsb, reflects the VC's costs of risk-sharing with the risk-averse EN. We refer to λs as the *price of risk*.

Return on investment. The "return on investment" term, $Kc(b)$, reflects the VC's expected return as a result of her investment and the EN's effort.

In the next two sections we show how the interactions among these three components determine how the equilibrium changes with market conditions.

4.4 Comparisons with Observed Contractual Structures

By (16), the change in the EN's stake in the project over any period $[t, t + dt]$ has a "fixed" component $a_t^* V_t dt$ and a "risky" component $b_t^* dV_t$. Because the termination value process grows in expectation if the project is continued, and the VC has the bargaining power, the parameter a_t^* is generally negative. Hence, the component $a_t^* V_t dt$ is similar to a "debt" or "dividend" payment made by the EN, while the component $b_t^* dV_t$ is the "equity" portion of the EN's compensation over the period. Recall, however, that all payoffs occur upon termination so that no payments are actually made by the EN prior to termination. The portion $\int_0^\tau a_t^* V_t dt$ of the EN's termination payoff could be viewed as a *cumulative* debt or dividend payment from the EN to the VC fund while the portion $\int_0^\tau b_t^* dV_t$ is the cumulative outcome of the changes in the EN's equity stake over each period.

Because the VC fund's stake in the project at any date t is $V_t - P_t$, the VC fund's payoff at termination is $-\int_0^\tau a_t^* V_t dt + \int_0^\tau (1 - b_t^*) dV_t$. The VC fund, therefore, holds a contract that has "debt" as well as "equity" components. These features of the optimal contract are consistent with data on observed VC contracts reported by Sahlman (1990) and Kaplan and Stromberg (2003). They document that the most commonly observed security held by VCs is preferred stock in which VC investors hold a claim to a preferred dividend stream (that could be deferred) as well as an equity claim to any residual value of the venture. The complex path-dependence of the VC fund's and EN's payoffs, however, implies that the equilibrium contract between the VC and the EN can only be implemented (or approximated) using combinations of different financial securities, which is also consistent with the evidence in Kaplan and Stromberg (2003).

Because the VC has the bargaining power, we observe that the VC fund's claim to the firm's payoffs ensures that it recovers the cumulative investment $\int_0^\tau c_t^* V_t dt$ with very high probability. This prediction is also consistent with empirical evidence that the claims of VC investors in liquidation are at least as large as their original investments (see Section 3.4 of Kaplan and Stromberg, 2003). There is, however, a not insignificant probability that the VC fund does not recover its cumulative investment, especially if the project is terminated early because of poor intermediate realizations of the termination value. This is consistent with the evidence in Cochrane (2005) that a significant proportion (about nine percent) of VC projects fail to return their investments.

Another salient feature of venture capital agreements is the *vesting* of the EN's stake in the firm over time. In our framework, if the EN terminates the agreement at some date $s < \tau$, his payoff is $P_s = \int_0^s [a_t^* V_t dt + b_t^* dV_t]$, which is lower, in expectation, than his payoff P_τ if he were to continue the relationship until the optimal termination time τ.

Broadly, the above discussion shows that the equilibrium contract between the VC and the EN has many of the features observed in actual venture capital contractual structures. Since there is a lack of more precise data on actual contracts, in particular, more detailed information on the functional forms of actual payoffs, it is not possible to establish tighter links between the optimal contracts predicted by our theory and observed contracts.

5 Equilibrium Dynamics

In this section, we investigate the properties of the equilibrium contract between the VC and the EN. Before analyzing the general scenario with asymmetric beliefs, we briefly discuss two *benchmark* scenarios.

5.1 Symmetric Attitudes towards Risk and Symmetric Beliefs about Project Quality—No Agency

In this scenario, the VC and the EN are both risk-neutral and have symmetric beliefs about the project's quality. Therefore, $\lambda = 0$ and $\Delta_t = 0$ for all t. It follows that the economic rent from the EN's optimism and the cost of risk components the VC's objective function (34) are zero. The third component, the return on investment, is always maximized at $b = 1$ (Proposition 1). Therefore, the equilibrium levels of pay performance sensitivity, the VC's investment rate and the EN's effort are constant through time, and the VC's investment rate is at its highest possible level. These results follow from the fact that as the VC and the EN have symmetric attitudes towards risk and symmetric beliefs, they effectively function as a monolithic agent. Moreover, the risk-neutrality of the VC/EN implies that risk (intrinsic and transient) of the project does not affect the investment rate, the EN's contract or his effort.

5.2 Symmetric Beliefs

In this scenario, the VC's objective function

$$F_t(b) = F(b) := -\lambda s b + K c(b) \tag{36}$$

is independent of time. It is also strictly concave by Proposition (1). The time paths of pay performance sensitivity, investment and effort are all constant; we let b_p^*, c_p^* and η_p^* denote the corresponding equilibrium values.

By Proposition 1 the optimal investment function achieves it maximum at $b = 1$, which implies that $c'(1) = 0$. It then follows from (36) that $F'(1) < 0$. Since $F'(b_p^*) = 0$, the strict concavity of $F(\cdot)$ now guarantees that $b_p^* < 1$, and therefore both c_p^* and η_p^* are less than the investment rate and effort levels in the "no agency" scenario where the VC and the EN are both risk-neutral and have symmetric beliefs about project quality.

5.3 Imperfect Information and Asymmetric Beliefs—The Actual Scenario

In the actual scenario, the VC's objective function may be expressed as

$$F_t(b) = \frac{\Delta_0}{\sigma_0^2}\sigma_t^2 b + F(b). \tag{37}$$

Since $\sigma_t \to 0$, it follows from the Theorem of the Maximum that $b_t^* \to b_p^*$, and thus $(c_t^*, \eta_t^*) \to (c_p^*, \eta_p^*)$ by continuity where (b_p^*, c_p^*, η_p^*) are the equilibrium pay-performance sensitivity, investment rate, and effort in the benchmark scenario with symmetric beliefs discussed in the previous subsection. We now characterize the manner in which these economic variables converge to their asymptotic values (conditional on the project's continuation).[10]

Theorem 2 (The Dynamics of the Equilibrium)
The EN's pay-performance sensitivity, b_t^*, the VC's investment rate, c_t^*, and the EN's effort, η_t^*, all decrease monotonically with t and respectively approach b_p^*, c_p^* and η_p^* as $t \to \infty$.

The results of Theorem 2 hinge on the interplay among the value-enhancing effort by the EN that is positively affected by his optimism, the costs of risk-sharing due to the EN's risk aversion that are negatively affected by the project's intrinsic risk, and the effect of both the VC's physical capital investment and the EN's effort on output. The passage of time lowers the degree of asymmetry in beliefs as successive project realizations cause the EN to revise his initial assessment of project quality. Since the EN is optimistic, he is willing to accept a greater portion of the project's risk so that his pay performance sensitivity and effort are initially high. The negative effect of the evolution of time on the EN's optimism, however, causes the EN's pay performance sensitivity and effort to decline over time. The decreasing effort of the EN makes it optimal for the VC to also lower her capital investments.

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[10] In Section 7, we show that the termination time is a random stopping time.

6 Sensitivity of Equilibrium Dynamics

Theorem 3 below characterizes how the equilibrium paths of pay performance sensitivity, investment rates and EN's effort are affected by changes to the "risk" parameters (e.g. the transient risk, σ_0^2, the intrinsic risk, s^2, and the EN's cost of risk, λ), the initial degree of asymmetry of beliefs, Δ_0 and the EN's cost of effort, k.

Theorem 3 (Sensitivity of Equilibrium Dynamics)
The paths of the EN's pay performance sensitivity, the VC's investment rates and the EN's effort are each pointwise (a) decreasing functions of the EN's cost of risk; (b) decreasing functions of the initial transient risk; (c) decreasing functions of the intrinsic risk; (d) increasing functions of the initial degree of asymmetry of beliefs; and (e) decreasing functions of the EN's cost of effort.

The EN's pay performance sensitivity declines with his cost of risk because an increase in the EN's cost of bearing risk increases the costs of risk-sharing between the VC and the EN. An increase in the transient risk lowers the degree of asymmetry in beliefs at each date because the "signal to noise ratio" is increased so that the EN "learns faster". Hence, the economic rents to the VC in each period from the EN's optimism are lowered relative to the costs of risk-sharing so that the EN's pay-performance sensitivity declines. An increase in the intrinsic risk, on the other hand, increases the degree of asymmetry in beliefs at each date because the EN "learns more slowly" but also increases the costs of risk-sharing. Under Assumption (2), the costs of risk-sharing outweigh the benefits of the EN's optimism so that the EN's pay-performance sensitivity also decreases with intrinsic risk. As a consequence, the investment rates and EN efforts also decline.

An increase in the EN's optimism leads to a corresponding increase in the economic rents to the VC in each period. The VC exploits this in each period by increasing the pay-performance sensitivity and investment, thereby leading to an increase in effort by the EN. Under Assumption (1), the economic rents that the VC can potentially capture due to the EN's exaggerated assessment of project quality are low compared with the costs of risk-sharing and inducing effort from the EN. Therefore, as the EN's cost of effort increases, the VC lowers the EN's pay performance sensitivity as well as her own investment in the project in each period.

7 Project Duration

In this section, we investigate the optimal termination decision of the VC, which determines the project's duration. We begin by describing the optimal termination policy of the VC. At any date t, we show that the VC's continuation value is an increasing, continuous function of her current assessment, μ_t^*, of the project's quality. Since the VC continues the project if and only if her continuation value is nonnegative, there exists a trigger level at each date such that she continues the project if and only if her current assessment of the project's quality exceeds the trigger.

Proposition 2

The optimal stopping policy for the VC is a trigger policy: there exist μ_t^* *such that the VC terminates the project only if* $\mu_t^{VC} < \mu_t^*$.

Let $Y_t^* dt := (c_t^{*\alpha} \eta_t^{*\beta} - 0.5 s^2 - \ell_t) dt$. Since $d \ln V_t = Y_t^* dt + \xi_t dt$ by (2), it follows that

$$\ln V_t - \ln V_0 = \int_0^t d \ln V_u = \Big(\int_0^t Y_u^* du \Big) + \Big(\int_0^t \xi_u du \Big).$$

Given the formula for μ_t given in (4), we may conclude that

$$\mu_t \geq \mu_t^* \text{ if and only if } V_t \geq V_t^*,$$

where

$$V_t^* := \exp \left[V_0 + \Big(\int_0^t Y_u^* du \Big) + \frac{(s^2 + t\sigma_0^2)\mu_t^* - s^2 \mu_0}{\sigma_0^2} \right].$$

The process $\{V_t^*\}$ represent the *performance targets* that the project must meet at each date to ensure the continuation of the relationship. Thus, either the μ_t^* or the V_t^* may be used to define the VC's termination policy; the performance targets are more commonly used in practice.

An increase in the EN's initial degree of optimism about project quality increases the rents that the VC is able to extract by exploiting the EN's optimism thereby increasing her expected continuation value at each point in time. Hence, it is optimal for the VC to prolong the project's duration. An increase in the EN's cost of risk or cost of effort, however, increases the costs of risk-sharing for the VC, thereby lowering her continuation value at each point in time. Hence, the VC terminates the project earlier. The following result summarizes the effect of the EN's initial assessment of project quality, his cost of risk, and his cost of effort on the duration of the project.

Proposition 3

The project duration τ *increases with the initial degree of asymmetry in beliefs, decreases with the EN's cost of risk, and decreases with the EN's cost of effort.*

The following result establishes that the project is terminated in finite time almost surely. The result ensures that our assumption of a finite time horizon for the VC-EN relationship does not entail a significant loss of generality.

Proposition 4

For any $\delta > 0$ *there exists a* $T > 0$ *such that if the maximum possible time horizon is* $T' \geq T$*, then the project duration is strictly less than* T *with probability greater than* $1 - \delta$.

8 Calibration and Estimation

We have developed a structural model of venture capital investment and described static and dynamic properties of the equilibrium. In this section we determine the baseline values of the parameters of the model by matching the model to empirical evidence. In the next section we use the matched model to assess the impact of asymmetric beliefs and agency conflicts on venture capital relationships.

The parameters of the model can be classified into two groups: those whose baseline values can be directly set using guidance from previous empirical research, and those whose values are indirectly determined by matching statistics predicted by the model to empirical evidence. With respect to the direct parameters, we assume a production technology with constant returns to scale so that $\beta = 1 - \alpha$. We assume that the function $l(t)$, which determines the losses in each period, has the quadratic form $l(t) = l_1 t^2$. Cochrane (2005) finds that the average return on venture capital investment in his sample is 15%. We, therefore, set the benchmark return R_b to 15%. Finally, Cochrane (2005) reports an average risk-free rate of 0.068 for his sample so that we set the risk-free rate r to 0.068.

There are nine parameters of the model whose values we cannot directly set using guidance from previous empirical research. These parameters are the capital intensity α of the production function, the initial transient risk σ_0^2, the intrinsic risk s^2, the "level" of losses in each period l_1, the VC's initial mean assessment of project quality μ_0^{VC}, the initial degree of asymmetry in beliefs Δ_0, the EN's cost of risk λ, the EN's "level" of disutility of effort k, and the disutility of effort exponent γ.

Table 4 of Cochrane (2005) reports data on the risks and returns of VC projects in rounds 1, 2, 3, and 4, as well as the average number of rounds of financing. We indirectly infer the above nine parameters by simulating the model and matching the *predicted* round-by-round risks and expected returns of a VC project and the mean number of rounds of financing to their corresponding *observed* values in Cochrane (2005).

In our numerical analysis, we use a discrete-time approximation of the continuous-time model, where the time interval between successive dates is the length of a round of financing.[11] As Gompers (1995) reports that the mean length of a round of VC financing is approximately one year, we set the time interval between successive dates to one year. In our framework, the analogues of the

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[11] In reality, venture capitalists also stage capital investments *within* a round of financing. We abstract from the staging of investments within rounds in our framework.

round-by-round returns of a VC project in Cochrane (2005) are

$$\ln\left[(V_t - c_{t-1}V_{t-1})/V_{t-1}\right], \quad t = 1, 2, 3, 4.$$

We use nine statistics to estimate the values of nine parameters.[12]

Estimated Parameters, Observed and Predicted Statistics

The numerical implementation proceeds as follows. As explained in Section 7, the key stochastic variable in the model is the VC's current assessment of project quality, μ_t^{VC}, which determines whether the relationship is continued or terminated at any date t. Therefore, we directly model the evolution of μ_t^{VC}, which is described by (5). In the first stage, we approximate the evolution of μ_t^{VC} using a discrete lattice and derive the termination triggers μ_t^*. In the second stage, given the triggers obtained from the first step, we use Monte Carlo simulation to model the evolution of μ_t^{VC} and to obtain the key output variables of interest.

For each candidate vector of parameters, we compute the values of the aforementioned nine statistics predicted by our model. We then use an optimization routine in MATLAB to find the set of parameter values for which the statistics predicted by our model most closely match their observed values. The measure of distance is the sum of the squares of the distances of the predicted values from the observed values.[13] Table 1 shows the baseline values of the model parameters obtained after estimation. Table 2 compares the predicted values to their observed counterparts. The model is able to match the observed statistics quite well.

Out-of-Sample Predictions

Using a dataset different from that of Cochrane (2005), Sahlman (1990) reports statistics on the distribution of terminal value from venture capital investments. To investigate the "out-of-sample" predictive ability of the model, we set the parameter values to their baseline values in Table 1 and compare the observed values of the statistics reported in Sahlman (1990) to their predicted values.

In Figure 1 of Sahlman (1990), he reports that 2% of total firm value is attributable to those firms whose return was negative, 23% of the total firm value is attributable to those firms whose return was between zero and five times the amount invested, and the remaining 75% of total firm value is attributable from those firms whose return exceeded five times the amount invested. Let $Inv := V_0 + \sum_{t=0}^{\tau} c_t^* V_t$ denote the firm's cumulative investment and let $Ret := V_\tau / Inv$ represent

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[12] In Table 3, Cochrane (2005) reports the risks and returns from each round to eventual IPO or acquisition. These statistics are not appropriate for our estimation because our model does not distinguish between the various ways in which a VC project could be terminated: IPO, acquisition, liquidation, etcetera.

[13] The details of the numerical implementation are available upon request.

the firm's return. In our model, these statistics are the ratios

$$\frac{E[V_\tau \, 1(Ret \in \mathcal{I}_j)]}{E[Inv \, 1(Ret \in \mathcal{I}_j)]}, \; j = 1, 2, 3,$$

where $\mathcal{I}_1 = (-\infty, 0]$, $\mathcal{I}_2 = (0, 5]$, $\mathcal{I}_3 = (5, \infty)$. Our model predicts corresponding values of 1.44%, 22.40% and 76.16% for the Sahlman (1990) statistics, which compare very well with the observed values 2%, 23% and 75%, respectively.

9 Numerical Results

In the numerical analyses to follow, we compare the *actual scenario* in which there are asymmetric beliefs and agency conflicts with the two benchmark scenarios discussed in Section 5, the *no agency* scenario, in which the VC and EN are both risk-neutral and have symmetric beliefs, and the *symmetric beliefs* scenario, in which beliefs are symmetric, but the EN's cost of risk is nonzero.

9.1 Output Variables

We compute several key output variables in each of these scenarios.[14] We interpret the benchmark return on assets R_b as the project's *cost of capital.* The "rational expectations" value of the firm at date zero, which is the expected total payoffs to the firm less the incremental capital investments discounted at the rate R_b, is

$$Firm\ Value = E_0^{VC}\Big[e^{-R_b \tau} V_\tau - \sum_{t=0}^{\tau-1} e^{-R_b t} c_t V_t\Big]. \tag{38}$$

The rational expectations value of the VC fund's stake in the firm at date zero is

$$VC\ Value = \text{Firm Value} - E_0^{VC}\Big[e^{-R_b \tau} P_\tau\Big], \tag{39}$$

which is the firm value less the termination payoff to the EN discounted at the rate R_b. We also compute and report the cumulative total investment in the project,

$$Investment = E_0^{VC}\Big[\sum_{t=0}^{\tau-1} c_t V_t\Big]. \tag{40}$$

The expectations in (38)-(40) are with respect to the VC's beliefs about the distribution of project quality, which are assumed to be correct. The termination value process evolves as in (1) with the contractual parameters, (a^*, b^*, c^*), the EN's effort, η^*, and the performance targets, V^*, set to their equilibrium values for the specific economic scenario (no agency, symmetric beliefs or actual) being analyzed. When necessary, we indicate the economic scenario with a superscript.

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[14] Keep in mind that the time period between successive rounds is set to one year in our discrete time approximation.

To measure the cost of agency and the benefit of EN optimism we define:

$$\begin{aligned} \textit{Agency Cost} &:= \frac{\text{Firm Value}^{\text{no agency}} - \text{Firm Value}^{\text{symmetric beliefs}}}{\text{Firm Value}^{\text{no agency}}} && (41) \\ \textit{Actual Cost} &:= \frac{\text{Firm Value}^{\text{no agency}} - \text{Firm Value}^{\text{actual}}}{\text{Firm Value}^{\text{no agency}}} && (42) \\ \textit{Benefit of Optimism} &:= \frac{\text{Firm Value}^{\text{actual}} - \text{Firm Value}^{\text{symmetric beliefs}}}{\text{Firm Value}^{\text{no agency}} - \text{Firm Value}^{\text{symmetric beliefs}}} . && (43) \end{aligned}$$

We also compute the above statistics when 'VC value' defined in (39) is substituted for firm value.

9.2 Baseline Analysis

Table 3 reports the firm value, VC value, and investment as defined above for the baseline model. We see that EN optimism significantly mitigates the deadweight agency costs of risk-sharing between the VC and the EN. When evaluated in terms of firm value, the agency cost is 58%, the actual cost is 52%, which implies that the benefit of EN optimism is 12%. With respect to the VC value, the agency cost is 28%, the actual cost is 17%, which results in the benefit of EN optimism being 37%. Hence, while EN optimism has a positive effect on firm value, it is significantly more beneficial to the VC. The VC exploits the EN's optimism to her advantage and disproportionately increases her share of the resulting surplus.

Table 3 also displays the distribution of the project's duration. Here, $p_i^* := Pr\{\tau = i + 1\}$. The project has a significantly greater probability (29.8%) of being terminated at the end of one year compared with the no agency scenario (16.3%). The benefits of EN optimism are, however, reflected in the fact that the project is more likely to last longer and, therefore, generate more value compared with the benchmark scenario in which beliefs are symmetric.

Table 4 reports the EN's pay-performance sensitivities, the VC's investment rates, the termination probabilities and the dollar investments $\mathcal{I}_i := c_i^* V_i$ (conditional on continuation) for the first four rounds. Consistent with Theorem 2, the EN's pay-performance sensitivity and the VC's investment rate decline over time. The EN's pay-performance sensitivity decreases sharply across the four periods, while the decrease in the VC's investment rate is gradual. Successive capital infusions by the VC, therefore, rapidly reduce the EN's relative stake in the firm.

9.3 Comparative Statics

We now carry out several "comparative statics" analyses by varying parameters about their baseline values in Table 1. We focus on exploring the comparative statics with respect to four parameters of interest, namely, the physical capital intensity of the production function, α, the initial transient risk, σ_0^2, the intrinsic risk, s^2, and the initial degree of asymmetry in beliefs, Δ_0.

Table 5 reports the expected project duration, firm values, VC values and investments for the actual scenario and the two benchmark scenarios. The table also reports the corresponding agency costs, actual costs and benefits of EN optimism as defined in (43). Table 6 shows the corresponding variations in the dynamics of the VC-EN contract.

The Effect of Human and Physical Capital Intensity

From Table 5, we observe that in the actual and benchmark scenarios, the expected project duration, firm value, VC value, and the total investment all decline with the physical capital intensity of the production function. In our numerical analyses, we assume a constant returns-to-scale production function so that $\alpha + \beta = 1$. Hence, as the physical capital intensity increases, the human capital intensity decreases, which reduces the relative contribution of the EN's effort to output.

As shown by Table 6, the VC's proportional investment rate in the project in any round (conditional on continuation) increases as the physical capital intensity increases. However, the VC's dollar investment in any round i is $c^*_{i-1}V_{i-1}$. As the physical capital intensity increases, the lower surplus generated by the EN's effort causes the termination value process to decline path-wise and the termination probabilities increase, i.e., the project terminates "earlier". Hence, even though the proportional investment rates conditional on continuation increase with the physical capital intensity, the total investment over the duration of the project declines. Moreover, consistent with the fact that the relative contribution of the EN's effort declines with the physical capital intensity, the EN's pay-performance sensitivities in each round decline as shown by Table 6.

Because the relative contribution of the EN's effort to output declines with the physical capital intensity of the production technology, the deadweight agency costs of risk-sharing between the VC and the risk-averse EN also decline. Interestingly, we observe from Table 5 that for relatively high physical capital intensities (e.g. $\alpha = 0.45$), the VC value in the actual scenario could *exceed* the VC value in the no agency scenario. In other words, it is possible for the EN optimism to more than completely offset the agency costs borne by the VC.

The above results lead to the following testable implications:

Testable Implications 1

a) The value of the firm, the payoff to the VC, the total investment, and the expected duration decrease with the physical capital intensity of the project and increase with its human capital intensity.

b) The VC's investment in any round as a proportion of the termination value of the project decreases with the project's physical capital intensity and increases with its human capital intensity.

c) The EN's pay-performance sensitivity in any round decreases with the project's physical capital intensity and increases with its human capital intensity.

The Effect of Technical Risk

An increase in the transient risk has two opposite effects on firm value, VC value and total investment. On the one hand, the EN "learns more quickly" due to an increase in the "signal to noise" ratio (see 7). The faster convergence of the EN's beliefs lowers the economic rents that the VC could extract from the EN's optimism, which has a negative effect. On the other hand, it increases the likelihood of both "high" and "low" realizations of project quality. Since the VC also learns faster about the true project quality and can limit her downside by terminating the relationship if intermediate signals of project quality are sufficiently poor, the option value of the project increases, which has a positive effect. As we observe from Table 5, the positive effect of the option value dominates the negative effect on the power of incentives to the EN, thereby leading to a sharp increase in the firm value, VC value and investment in the actual and benchmark scenarios.

The positive effect of the option value of continuing the relationship is further illustrated in Table 6, which shows that the probability of the project being continued beyond the third round dramatically increases. (It also shows that with a higher likelihood of a low realization, the probability of terminating at the end of round 1 increases.)

For high values of the transient risk, Table 5 shows that the firm value is over forty times the termination value V_0 at date zero in the actual scenario (recall that we normalize $V_0 = 1$). Consistent with the evidence in Cochrane (2005), this reflects the highly skewed nature of the payoffs from venture capital investments and the tremendous value that could be generated by the presence of uncertainty about the quality of innovative ventures. The sharp increase in the VC value with the project's transient risk, or the degree of uncertainty about its quality, is consistent with empirical evidence that VCs have significant incentives to finance highly innovative projects compared with mature or "imitating" projects because there is greater uncertainty about the quality of innovative projects (for example, Hellman and Puri, 2000).

The above results lead to the following testable implications:

Testable Implications 2

a) The value of the firm, the payoff to the VC and the total investment all increase with the project's initial transient risk.

b) The EN's pay-performance sensitivity in any round declines with the initial transient risk.

The Effect of Intrinsic Risk

Thus far, the benchmark return or the cost of capital R_b is fixed at its baseline value 15%. To explore the effects of varying the intrinsic risk, however, we must incorporate the fact that the cost of capital also varies with the intrinsic risk. For concreteness, we assume that the benchmark return is an affine function of the intrinsic risk, that is,

$$R_b(s) := r + \frac{\hat{R}_b - R_f}{\hat{s}}\, s, \tag{44}$$

where $\hat{R}_b$ and $\hat{s}$ denote the values for the benchmark return and the intrinsic risk reported in Table 1 and r is the average risk-free rate 0.068 reported in Cochrane (2005).

The effects of intrinsic risk are not *a priori* obvious because an increase in intrinsic risk has several competing effects on firm value, the VC value, and investment. First, in contrast to the effect of transient risk, an increase in the intrinsic risk causes the EN "learns more slowly" because the "signal to noise" ratio decreases (see 7). The slower convergence of the EN's beliefs has a positive effect on the rents the VC is able to extract from the EN's optimism. Second, an increase in the intrinsic risk increases the costs of risk sharing between the VC and the EN, which has a negative effect on firm value, the VC value, and investment. Third, the benchmark return $R_b(s)$ (the "hurdle rate" or "cost of capital") decreases with intrinsic risk, which has a negative effect on firm value and the VC value. Fourth, the effect of intrinsic risk on the option value of continuing the relationship is *non-monotonic.* The standard deviation of the assessment of project quality, σ_t^μ given by (6) decreases with the intrinsic risk before time $t(s) := (s/\sigma_0)^2$ and increases with the intrinsic risk after this time.[15] Since $t(s)$ increases with s, an increase in s^2 delays the benefits of the option value of continuing the relationship.

We observe from Table 5 that the expected duration, the firm value, the VC value, and the investment, all decline with the project's intrinsic risk in the actual and benchmark scenarios. The increase in the cost of risk, the lower economic rents and the higher cost of capital outweigh the potential increases in the option value. These results lead to the following testable implications:

Testable Implications 3

a) The value of the firm, the payoff to the VC, the total investment, and the expected project duration all decrease with the project's intrinsic risk.

b) The EN's pay-performance sensitivity in any round decreases with the project's intrinsic risk.

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[15] Follows directly from $d\sigma_t^\mu/ds = \sigma_0^2(t\sigma_0^2 - s^2)/(s^2 + t\sigma_0^2)^2$.

The Effect of the Degree of Asymmetry in Beliefs

The increase in firm value, the VC value, the project's duration, and investment with the degree of asymmetry in beliefs shown by Table 5 illustrate the positive effects of the EN's optimism. An increase in the degree of asymmetry in beliefs mitigates the costs of risk sharing between the VC and the EN thereby enhancing the power of incentives that could be provided by the EN. As shown by Table 6, the EN's pay-performance sensitivity in any round increases with the degree of asymmetry in beliefs. These results lead to the following testable implications:

Testable Implications 4

a) The value of the firm, the payoff to the VC, the total investment, and the expected project duration all increase with the initial degree of asymmetry in beliefs about project quality.

b) The EN's pay-performance sensitivity in any round increases with the initial degree of asymmetry in beliefs about project quality.

Prior literature reports empirical evidence that is indirectly consistent with the implication a) above. The positive effects of EN optimism on firm value are consistent with the empirical evidence reported in Gelderen, Thurik and Bosma (2005). The negative relation between duration and the degree of EN optimism is consistent with the evidence in Kaplan and Stromberg (2003) that experienced entrepreneurs, who are likely to have more realistic beliefs, receive fewer rounds of financing.

9.4 Implied Discount Rates

Sahlman (1990) reports that VC's typically adjust for EN optimism by discounting EN projections at high discount rates. To assess the magnitude of such adjustments, we calculate the *implied discount rate (IDR).* The VC "simulates" how the termination value process would proceed under the EN's initial beliefs about project quality. Since her beliefs have not changed, the equilibrium contract, (a^*, b^*, c^*), the EN's effort, η^*, and the performance targets, V^*, all remain unchanged. However, the signals of project quality now follow the EN's distribution. The implied discount rate is the rate at which the VC would have to discount her terminal payoff and investments in this new scenario to match her true beliefs about firm value and VC value defined in (38) and (39), respectively. More precisely, the implied discount rates β_F, β_{VC} solve:

$$E_0\Big[e^{-\beta_F \tau} V_\tau - \sum_{t=0}^{\tau-1} e^{-\beta_F t} c_t V_t \;\Big|\; \Theta \sim N(\mu_0^{EN}, \sigma_0^2)\Big] \;=\; Firm\,Value \tag{45}$$

$$E_0\Big[e^{-\beta_{VC} \tau} (V_\tau - P_\tau) - \sum_{t=0}^{\tau-1} e^{-\beta_{VC} t} c_t V_t \;\Big|\; \Theta \sim N(\mu_0^{EN}, \sigma_0^2)\Big] \;=\; VC\,Value. \tag{46}$$

Tables 7 and 8 report the IDR's for varying values of $\alpha, \sigma_0^2, s^2, \Delta_0$. Except for the results where Δ_0 is varied, all the IDR's lie between 0.475 and 0.525. These values lie in the "midpoint" of the range 0.40-0.60 for the IDR's reported in Table 6 of Sahlman (1990). Over a wide range of parameter settings, we see that the IDR's are all remarkably stable. Not surprisingly, the variation of the IDR's with Δ_0 is approximately linear.

Cochrane (2005) comments that the discrepancy between the high discount rates used by VCs and the lower ex post returns of VC investments is an enduring puzzle of the venture capital literature. Through the results of Tables 7 and 8, our study provides a rationale for these findings. Note that the firm value and the VC value according to the VC's beliefs, which are given by (38) and (39), respectively, are computed using a reasonable discount rate of 15%, which is the average return on VC projects reported in Cochrane (2005). The implied discount rates are much higher because they adjust for the EN's optimistic projections of the project's payoffs.

The fact that our model, which is calibrated to the risk and return data in Cochrane (2005), generates implied discount rates consistent with those reported in Sahlman (1990) strongly suggests that entrepreneurial optimism explains these findings. Moreover, the numerical results suggest that an IDR of 50% is a remarkably good rule-of-thumb to value venture capital projects.

10 Conclusions

We develop a dynamic, structural model to examine the effects of the interactions among risk, imperfect information, agency conflicts, and asymmetric beliefs on the characteristics of venture capital relationships. We show that the presence of entrepreneurial optimism is a key driving force of the economic value generated by VC relationships and explains why discount rates used by VC are much higher than discount rates predicted by standard asset pricing theories.

Our principal-agent framework incorporates several key features of VC relationships, namely, the presence of high levels of intrinsic and transient risk and resulting agency conflicts, imperfect information and asymmetric beliefs about project quality, and the importance of staged investment and dynamic contracting in mitigating potential inefficiencies arising from these market imperfections. Consistent with observed contractual structures, the optimal dynamic contract between the VC and EN has a "debt" and "equity" component. The EN's stake in the project vests over time, and the VC's claim allows her to recover her investments in the project with high probability.

The EN's pay-performance sensitivities and the VC's investment rates decline over time. The pay-performance sensitivity and investment rate paths decline with risk and increase with the initial degree of asymmetry in beliefs. The duration, firm value, and the VC value all increase with the degree of asymmetry in beliefs about project quality. VCs, therefore, have significant incentives to encourage or "feed" entrepreneur optimism and exploit it to their advantage, and these incentive also have beneficial effects on project values. The interplay among the characteristics of the underlying project—specifically, its intrinsic and transient risk and the degree of asymmetry in beliefs about its quality—leads to significant heterogeneity in contractual structures and the patterns of staged VC investment, which is consistent with empirical evidence. The intrinsic and transient risks of projects have opposite effects on the durations and economic values of VC relationships. The value of the project and the expected payoff to the VC are actually enhanced when there is greater noise in the perception of project quality.

The tractability of the model suggests that it could be useful as a tool to value risky projects when agents have asymmetric beliefs and have conflicts of interest. Although we focus on venture capital investment, our framework is more generally applicable to dynamic principal-agent settings with double-sided moral hazard, risk, imperfect information, and asymmetric beliefs. For example, our framework could be applied to study the relationship between the shareholders of a firm and its manager or employees, the financing of research and development, and delegated portfolio management (mutual funds, hedge funds). We explore these applications in future research.

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Appendix A

The Relation Between Outside Investors and the VC

Let $\mathcal{A}_t$ represent the total assets under management at date t, which we derive endogenously in the following. The fund charges a constant proportional management fee, $f > 0$, and has quadratic operating costs $C(\mathcal{A}_t) = M\mathcal{A}_t + N\mathcal{A}_t^2$. The expected expected excess return on assets *before operating costs and fees* is $E_t[R_{t+dt} - R_b]$, where R_{t+dt} is given by (8). Hence, the expected excess return on assets of the fund *net of operating costs and fees* or the expected net excess return is given by

$$\frac{\mathcal{A}_t E_t[R_{t+dt} - R_b dt] - f\mathcal{A}_t dt - C(\mathcal{A}_t)dt}{\mathcal{A}_t}\ . \tag{47}$$

As in Berk and Green (2004), the supply of capital by outside investors is perfectly competitive so that in equilibrium investors allocate capital to the fund until the expected net excess return is zero. It follows directly from (47) that the assets under management are affine in the expected excess return. Since the VC's compensation is $fQ_t dt$, it is also affine in the expected excess return.

The Representation of the EN's Utility as a Recursive or Stochastic Differential Utility

We show that the EN's utility as described by (10) or (11) actually belong to the general class of *stochastic differential utilities* introduced by Duffie and Epstein (1992). If

$$U_t = E_t^{EN}\Big[e^{-r(\tau-t)}P_\tau - \int_t^\tau e^{-rs}k\eta_s^\gamma V_s ds)\Big] \tag{48}$$

is the EN's conditional expected utility at date t, then U_t satisfies the following *backward stochastic differential equation* or BSDE (see Ma and Yong, 1999):

$$\begin{aligned} dU_t &= (rU_t - k\eta_t^\gamma V_t)dt + Z_t dB_t^{EN} \\ &= (rU_t - k\eta_t^\gamma V_t - \lambda Z_t)dt + Z_t dB_t, \\ U_\tau &= P_\tau \end{aligned} \tag{49}$$

The second equality above follows from (13). The process Z_t is $\{\mathcal{F}_t\}$ adapted and the pair of processes (U, Z) is the solution of the BSDE (49). From (49), the EN's conditional expected utility U_t has the general recursive representation

$$\begin{aligned} U_t &= U_\tau + \int_t^\tau f(U_s, Z_s)ds - \int_t^\tau Z_s dB_s, \\ f(U_s, Z_s) &= -rU_s + k\eta_s^\gamma V_s + \lambda Z_s, \end{aligned} \tag{50}$$

where the function f is the *aggregator* (see Duffie and Epstein, 1992, Ma and Yong, 1999).

Appendix B: Proof of Theorem 1

A rigorous proof of Theorem 1 requires a precise interpretation of equation (1), which describes the evolution of the termination value process. As we explain below, instead of viewing the termination value process as a stochastic process that is affected by the VC's and EN's actions, we will consider the termination value process $V(\cdot)$ to be a *given* random process on a probability space with investment and effort altering the *probability distribution* of this process.

The Termination Value Process

We consider an underlying probability space $(\Omega, \mathcal{F})$ with probability measures Q^ℓ, $\ell \in \{VC, EN\}$, representing the VC's and EN's beliefs. Θ is a normal random variable with variance σ_0^2 and mean μ_0^ℓ under measure Q^ℓ and $\widehat{B}$ is a standard Brownian motion. The complete and augmented filtration of the probability space generated by the Brownian motion $\widehat{B}(\cdot)$ is denoted by $\{\mathcal{F}_t\}$. Consider the process $V(\cdot)$ defined by the stochastic differential equation

$$dV(t) = sV(t)d\widehat{B}(t), \tag{51}$$

where s^2 is the intrinsic risk of the project. We use the Girsanov transformation (see Oksendal, 2003) to obtain new probability measures on $(\Omega, \mathcal{F})$ such that the process $V(\cdot)$ evolves as in (1).

Suppose that $\eta(\cdot)$ and $c(\cdot)$ are strictly positive, square-integrable $\{\mathcal{F}_t\}$-measurable stochastic processes (under the measures Q^{VC} and Q^{EN}) defined on the time horizon $[0, T]$ describing the EN's choices of effort and the VC's choices of investments over time. Recall that $l(\cdot)$ is a deterministic process describing the operating costs of the firm. Define the processes

$$\begin{aligned}\zeta_{c,\eta}(t) := \exp\Big[&\int_0^t (\Theta + c(u)^\alpha \eta(u)^\beta - l(u) - \lambda s 1_{\ell=EN}) s^{-1} d\widehat{B}(u) - \\ &\frac{1}{2}\int_0^t (\Theta + c(u)^\alpha \eta(u)^\beta - l(u)^2 - \lambda^2 s^2 1_{\ell=EN}) s^{-2} du\Big]\end{aligned} \tag{52}$$

and

$$B_{c,\eta}^\ell(t) := \widehat{B}(t) - \int_0^t (\Theta + c(u)^\alpha \eta(u)^\beta - l(u) - \lambda s 1_{\ell=EN}) s^{-1} du \, . \tag{53}$$

The process $\zeta_{c,\eta}(\cdot)$ is a positive martingale.[16] Define the new measure $\Pi_{c,\eta}^\ell$ by

$$\frac{d\Pi_{c,\eta}^\ell}{dQ^\ell} = \zeta_{c,\eta}(T). \tag{54}$$

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[16] The processes are assumed to satisfy the Novikov condition (see Oksendal, 2003):

$$E^\ell \exp\Big[\frac{1}{2}\int_0^T (\Theta + c(u)^\alpha \eta(u)^\beta - \lambda s 1_{\ell=EN} - l(u))^2 s^{-2} du\Big] < \infty, \ \ell \in \{VC, EN\} \, .$$

Because the equilibrium investment and effort processes described in Theorem 1 are deterministic and Θ is a normal random variable, the Novikov condition is satisfied by these processes.

In fact, we do not need to assume that feasible (not necessarily optimal) investment and effort processes satisfy the Novikov condition for our analysis to be valid; we only require that they be square-integrable. In this case, the process $\zeta_{c,\eta}(\cdot)$ is only guaranteed to be a local martingale and the measure $\Pi_{c,\eta}^\ell$ is a finite measure, but not necessarily a

By Girsanov's theorem (see Oksendal, 2003), the process $B^{\ell}_{c,\eta}(\cdot)$ is a Brownian motion under the measure $\Pi^{\ell}_{c,\eta}$. Further, under this measure, the process $V(\cdot)$ evolves as

$$dV(t) = [\Theta + c(t)^{\alpha}\eta(t)^{\beta} - l(t) - \lambda s 1_{\ell=EN}]V(t)dt + sV(t)dB^{\ell}_{c,\eta}(t)\ . \tag{55}$$

Equation (55) describes the evolution of the termination value process under the actual probability and the EN's subjective valuation probability, and is identical to equations (1) and (12), but with the Brownian motion and the probability measures representing the VC's and EN's beliefs depending on the investment and effort processes. It is important to keep in mind that $V(\cdot)$ is a *fixed* process whose sample paths are not affected by investment and effort. Investment and effort, however, alter the probability distribution of the sample paths of $V(\cdot)$.

For future reference, we make an important observation. The process

$$dW^{\ell}_{c,\eta}(t) := s^{-1}[d\ln V(t) - (c(t)^{\alpha}\eta(t)^{\beta} - 0.5s^2 - \lambda s 1_{\ell=EN} - l(t))dt - \mu^{\ell}_t dt] \tag{56}$$

is an $\{\mathcal{F}_t\}$-Brownian motion with respect to the probability measure $\Pi^{\ell}_{c,\eta}$. Moreover, the complete and augmented filtration generated by this Brownian motion is $\{\mathcal{F}_t\}$. Recall that the EN's and VC's mean assessments of project quality Θ at date t, μ^{EN}_t, μ^{VC}_t in (2) and (4), do not depend on the effort and investment processes because they are observable.

Utility Related Processes

Let $\tau \leq T$ be an $\{\mathcal{F}_t\}$-stopping time denoting the termination time of the VC-EN relationship. Let $c(\cdot)$, $\eta(\cdot)$ and $\widehat{\eta}(\cdot)$ be strictly positive $\{\mathcal{F}_t\}$-adapted square-integrable processes on $[0,\tau]$.[17] A *contract* is represented by the triple $(P(\cdot), c(\cdot), \tau)$. The processes below are required in the sequel.

The *cumulative value process of the EN* is the EN's conditional valuation of his future payoffs at any date *including the sunk disutilities of prior effort* from a given contract $(P(\cdot), c(\cdot), \tau)$ when his effort choices over time are given by the process $\eta(\cdot)$. Formally, we define

$$\overline{U}_{P,c,\tau}(\eta(\cdot);t) := E^{EN}_{c,\eta}[P(\tau) - \int_0^{\tau} k\eta(u)^{\gamma}V(u)du]) \mid \mathcal{F}_t]\ . \tag{57}$$

Here, $E^{\ell}_{c,\eta}[\cdot \mid \mathcal{F}_t]; \ell \in \{VC, EN\}$ denotes conditional expectation at date t under the probability measure $\Pi^{\ell}_{c,\eta}$ defined in (54). For future reference, we note that the cumulative value process of the EN is a square-integrable $\{\mathcal{F}_t\}$-martingale under the measure $\Pi^{EN}_{c,\eta}$.

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probability measure. Our analysis, however, only requires that $\Pi^{\ell}_{c,\eta}$ be a finite measure. Since, as mentioned earlier, the Novikov condition is satisfied by the equilibrium investment and effort processes, the measure corresponding to the equilibrium processes is a probability measure.

[17] These processes are assumed to satisfy the Novikov condition—see footnote 16.

For a given contract $(P(\cdot), c(\cdot), \tau)$ and for a given effort process $\eta(\cdot)$, the *certainty equivalent process for the EN* is defined as

$$R_{P,c,\tau}(\eta(\cdot);t) := \overline{U}_{(P,c,\tau)}(\eta(\cdot);t) + \int_0^t k\eta(u)^\gamma V(u)du \ , \ t \in [0,\tau] \ . \tag{58}$$

At date t, $R_{P,c,\tau}(\eta(\cdot);t))$ equals the EN's expected utility for the remaining duration of the VC-EN relationship.

The *adjusted cumulative value process of the EN* represents the cumulative value process of the EN when he exerts effort $\eta(s); s \leq t$ and effort $\widehat{\eta}(s); s \geq t$. Formally, we define

$$Y_{P,c,\tau}(\eta(\cdot);t;\widehat{\eta}(\cdot)) := E^{EN}_{c,\eta}[P(\tau) - \int_0^t k\eta(u)^\gamma V(u)du - \int_t^\tau k\widehat{\eta}(u)^\gamma V(u)du]) \mid \mathcal{F}_t] \ . \tag{59}$$

The *EN's maximum conditional valuation process* represents the EN's maximum conditional valuation of his future payoffs at date t given that he has exerted effort $\eta(s); s \leq t$ and the contract is $(P(\cdot), c(\cdot), \tau)$. Formally, we define

$$Z_{P,c,\tau}(\eta(\cdot);t) := sup_{\widehat{\eta}(\cdot)} Y_{P,c,\tau}(\eta(\cdot);t;\widehat{\eta}(\cdot)) \ . \tag{60}$$

Implementation of a Given EN Effort Process

To simplify the subsequent notation, we drop the subscripts denoting the dependence of the processes defined in (57)-(60) on the contract $(P(\cdot), c(\cdot), \tau)$. A contract $(P(\cdot), c(\cdot), \tau)$ is said to *implement a given effort process* $\eta^*(\cdot)$ if and only if $P(0) = V(0)$ and, given the contract $(P(\cdot), c(\cdot), \tau)$, the EN's optimal effort choices are given by the process $\eta^*(\cdot)$. The following Lemma characterizes the contract $(P(\cdot), c(\cdot), \tau)$ that implements a given effort process $\eta^*(\cdot)$ of the EN.

Lemma 1

a) A contract $(P(\cdot), c(\cdot), \tau)$ implements $\eta^*(\cdot)$ only if $P(0) = V(0)$, $P(t) = R(\eta^*(\cdot);t)$ a.s., and the certainty equivalent process $R(\eta^*(\cdot);\cdot)$ satisfies the following stochastic differential equation:

$$dR(\eta^*(\cdot);t) = a(t)V(t)dt + b(t)dV(t) \tag{61}$$

where

$$b(t) = \frac{\gamma k}{\beta c(t)^\alpha}\eta^*(t)^{\frac{\gamma-\beta}{\beta}}, \tag{62}$$

$$a(t) := k\eta^*(t)^\gamma - b(t)\Big(c(t)^\alpha \eta^*(t)^\beta - \lambda s - l(t) + \mu_t^{EN}\Big). \tag{63}$$

b) The EN's continuation value defined in (14) is zero at each date.

Proof. a) When the certainty equivalent process $R(\eta^*(\cdot);t)$ corresponding to the given effort process $\eta^*(\cdot)$ evolves as in (61), we show below that the EN's optimal effort choices after any given date t coincide with the process $\eta^*(\cdot)$ regardless of his prior history of effort choices. In other words, the process $\eta^*(\cdot)$ solves (60) for any prior effort process $\eta(\cdot)$. Since the date t is arbitrary, it will then follow that the EN's optimal effort choices over the entire interval $[0, \tau)$ correspond to the process $\eta^*(\cdot)$ so that the contract $(P(\cdot), c(\cdot), \tau)$ implements the given effort process $\eta^*(\cdot)$.

By the principle of optimality of dynamic programming (Oksendal, 2003), the effort $\eta^*(t)$ is optimal for the EN at date t for any prior effort process $\eta(\cdot)$ only if [18]

$$\eta^*(t) = argmax_{\eta(t)} E_{c,\eta}[Z(\eta(\cdot); t + dt) - Z(\eta(\cdot); t) \mid \mathcal{F}_t] = argmax_{\eta(t)} E_{c,\eta}[dZ(\eta(\cdot); t) \mid \mathcal{F}_t] \ . \quad (64)$$

In what follows, we derive the infinitesimal change $dZ(\eta(\cdot); t)$ and then use (64) to establish the statements of the Lemma.

It follows from the definition (60) that

$$\begin{aligned} Z(\eta(\cdot); t) &= sup_{\eta'(\cdot)} Y(\eta(\cdot); t; \eta'(\cdot)) \\ &= sup_{\eta'(\cdot)} Y(\eta^*(\cdot); t; \eta'(\cdot)) + k \int_0^t (\eta(s)^\gamma - \eta^*(s)^\gamma) V(s) ds \\ &= Z(\eta^*(\cdot); t) + X(\eta(\cdot); t) \ , \end{aligned} \quad (65)$$

where we define the stochastic process

$$X(\eta(\cdot); t) := k \int_0^t [\eta(s)^\gamma - \eta^*(s)^\gamma] V(s) ds \ . \quad (66)$$

The second equality in (65) follows because the contract $P(.)$ only depends on the history of the termination value process, and because the EN's effort choices are observable, and so his prior effort choices over the interval $[0, t]$ do not affect his optimal effort choices over the interval $[t, \tau]$. It may be readily verified from (65) and (66) that

$$dZ(\eta(\cdot); t) = dZ(\eta^*(\cdot); t) + k(\eta(t)^\gamma - \eta^*(t)^\gamma) V(t) dt \ . \quad (67)$$

Since the process $\eta^*(\cdot)$ represents the EN's optimal effort choices by hypothesis, it follows that

$$Z(\eta^*(\cdot), t) = \overline{U}(\eta^*(\cdot), t), \quad (68)$$

and hence the process $Z(\eta^*(\cdot), \cdot)$ is a square-integrable $\{\mathcal{F}_t\}$-martingale under the measure Π^{EN}_{c,η^*}. It follows from (56) and the martingale representation theorem (see Oksendal, 2003) that there exists a square-integrable, $\{\mathcal{F}_t\}$-adapted process $\omega(\cdot)$ such that[19]

$$dZ(\eta^*(\cdot); t) = \omega(t) dW^{EN}_{c,\eta^*}(t) = \omega(t) s^{-1} [d \ln V(t) - (c(t)^\alpha \eta^*(t)^\beta - \lambda s - l(t)) dt - \mu^{EN}_t dt] \ . \quad (69)$$

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[18] Since the conditional expectation above only depends on the process $\eta(\cdot)$ prior to date t, which is an arbitrary process anyway, we avoid complicating the notation unnecessarily in (64) by using the same letter to denote a candidate (possibly sub-optimal) level of effort in the infinitesimal interval $[t, t + dt]$.

[19] Identity (69) is an *almost sure* relation that holds under all equivalent probability measures on the probability space. It is only under the measure Π^{EN}_{c,η^*} defined in (54) that the process $[dV(t) - (c(t)^\alpha \eta^*(t)^\beta - l(t)) dt - \mu^{EN}_t dt]$ is the increment of a Brownian motion.

Since the expectation in the dynamic programming equation (64) is taken under the measure $\Pi_{c,\eta}^{EN}$, it follows from (56) and (69) that $Z(\eta^*(\cdot);t)$ evolves under this measure as

$$dZ(\eta^*(\cdot);t) = \omega(t)s^{-1}c(t)^{\alpha}(\eta(t)^{\beta} - \eta^*(t)^{\beta})dt + \omega(t)dW_{c,\eta}^{EN}(t) \ . \tag{70}$$

Substituting (70) in (67) yields

$$dZ(\eta(\cdot);t) = \omega(t)s^{-1}c(t)^{\alpha}(\eta(t)^{\beta} - \eta^*(t)^{\beta}) + k(\eta(t)^{\gamma} - \eta^*(t)^{\gamma})V(t)dt + \omega(t)dW_{c,\eta}^{EN}(t) \ . \tag{71}$$

Having derived the requisite expression for $dZ(\eta(\cdot);t)$, we substitute it in (64) to obtain

$$\eta^*(t) = argmax_{\eta(t)}[\omega(t)s^{-1}c(t)^{\alpha}\eta(t)^{\beta} + k\eta(t)^{\gamma}V(t)] \ . \tag{72}$$

It then follows that the effort $\eta^*(t)$ is optimal over the interval $[t, t+dt]$ only if

$$\frac{\omega(t)}{V(t)} = -\frac{ks}{c(t)^{\alpha}}\ \frac{\gamma}{\beta}(\eta^*(t))^{\frac{\gamma-\beta}{\beta}} \ . \tag{73}$$

From the definition of the certainty equivalent process $R(\eta^*(\cdot);\cdot)$ in (58), and using (68), we have

$$R(\eta^*(\cdot);t) = Z(\eta^*(\cdot);t) + \int_0^t k\eta^*(u)^{\gamma}V(u)du \ . \tag{74}$$

Using Ito's Lemma, (69), (70), and (73), we obtain

$$dR(\eta^*(\cdot);t) = a(t)V(t)dt + b(t)dV(t), \tag{75}$$

where $b(t)$ and $a(t)$ are given by (62) and (63), respectively, as claimed.

Since the EN's maximum expected future utility from continuing the relationship at date $t \leq \tau$ is $-R(\eta^*(\cdot);t))$ and since his utility from terminating the relationship is $P(t)$, the EN continues the relationship until time τ only if $P(t) \leq R(\eta^*(\cdot);t)$ at each date t. Replacing t with the stopping time τ and $\eta(\cdot)$ with $\eta^*(\cdot)$ in (57) and (58), we see that $P(\tau) = R(\eta^*(\cdot);\tau)$. Hence, $P(t) = R(\eta^*(\cdot);t)$ for any time t such that $\tau = t$ has positive probability. We conclude that $P(t) = R(\eta^*(\cdot);t)$ almost surely. Since the EN owns the project at date 0, we must have $P(0) = V(0)$.

b) This result follows from the result of part a) that $P(t) = R(\eta^*(\cdot);t)$ and the fact that $R(\eta^*(\cdot);t) = E_t^{EN}R(\eta^*(\cdot);t+dt)$. ∎

As a consequence of Lemma 1, we can restrict consideration to contracts P where

$$dP(t) = a(t)V(t)dt + b(t)dV(t) \tag{76}$$

where $b(t) > 0$. Further, it follows from (62) that there is a one-one correspondence between the pay-performance sensitivity $b(t)$ and the EN's optimal effort $\eta(t)$ at date t.

Optimal Contract Choice by VC

By (62) and (63), a feasible contract is completely described by the investment process $c(\cdot)$, the processes $a(\cdot), b(\cdot)$ describing the fixed and proportional components of the EN's compensation and the termination time τ. Define

$$M_{a,b,c,\tau}(0) = E^{VC}_{c,\eta}\Big[(V(\tau) - P(\tau) - \int_0^\tau c(s)V(s)ds)\Big] \tag{77}$$

as the VC's expected future payoff at date 0 if she chooses a contract $(P(\cdot), c(\cdot), \tau) \equiv (a(\cdot), b(\cdot), c(\cdot), \tau)$. The VC's contract choice problem is then the following stochastic control problem

$$(a^*(\cdot), b^*(\cdot), c^*(\cdot), \tau^*) = argmax_{(a,b,c,\tau)} M_{a,b,c,\tau}(0). \tag{78}$$

Let the "state" of the system at any date t be described by the ordered pair (t, μ^{VC}_t). We first restrict consideration to *Markov controls* where $a(t), b(t), c(t)$ and the decision to terminate the relationship only depend on the current state (t, μ^{VC}_t). We derive the optimal Markov control policy. We then appeal to the *verification theorem* of dynamic programming (see Theorem 11.2.3 of Oksendal, 2003) to conclude that the optimal Markov control policy is, in fact, the optimal control policy over the entire space of admissible $\{\mathcal{F}_t\}$-adapted controls.

We note from (62) and (63) that the control $a(\cdot)$ is, in fact, determined by the controls $b(\cdot)$, $c(\cdot)$ and the state of the system. Hence, a Markov control policy is completely described by $(b(\cdot), c(\cdot), \tau)$. For simplicity, we abuse notation by denoting the VC's continuation value in state (t, μ^{VC}_t) from adopting the Markov control policy $(b(\cdot), c(\cdot), \tau)$ by

$$M_{b,c,\tau}(t, \mu^{VC}_t) = E^{VC}_{t;c,\eta}\Big[(V(\tau) - V(t)) - (P(\tau) - P(t)) - \int_t^\tau c(s)V(s)ds\Big], \tag{79}$$

where $\eta(\cdot)$ is determined by inverting (62). Let $M^*(t, \mu^{VC}_t)$ be the optimal continuation value within the space of Markov controls and $(b^*(\cdot), c^*(\cdot), \tau^*)$ be the optimal Markov control policy (we derive this policy in the following). Suppose that the VC deviates from the optimal policy over the infinitesimal time interval $[t, t+dt]$ by choosing the controls $(\widehat{b}(t), \widehat{c}(t))$, Let $\widehat{M}(t, \mu^{VC}_t)$ denote the VC's continuation value at date t under this deviated policy. It follows from (76) and (79) that

$$\widehat{M}(t, \mu^{VC}_t) = E^{VC}_{t;c,\eta}\Big[- \widehat{a}(t)V(t)dt + (1 - \widehat{b}(t))dV(t) - \widehat{c}(t)V(t)dt + M^*(t+dt, \mu^{VC}_{t+dt})\Big]. \tag{80}$$

By (55), we have

$$\widehat{M}(t, \mu^{VC}_t) = E^{VC}_{t;c,\eta}\Big[-\widehat{a}(t)V(t)dt + (1-\widehat{b}(t))[\mu^{VC}_t + \widehat{c}(t)^\alpha \widehat{\eta}(t)^\beta - l(t)]V(t)dt - \widehat{c}(t)V(t)dt + M^*(t+dt, \mu^{VC}_{t+dt})\Big], \tag{81}$$

where $\widehat{\eta}(t)$ is given by (62) with $\widehat{b}(t)$ replacing $b(t)$. Since the VC's investment and EN's effort are observable, the VC's assessment μ^{VC}_{t+dt} of project quality at date $t+dt$ is *independent* of the choices of controls $(\widehat{b}(t), \widehat{c}(t))$. Hence, the function $M^*(t+dt, \mu^{VC}_{t+dt})$ is also independent of these choices. By the principle of optimality of dynamic programming (see Fleming and Soner, 1992), the optimal controls $(b^*(t), c^*(t))$ at date t must maximize the "flow" term in (81), that is,

$$(b^*(t), c^*(t)) = argmax_{\widehat{b}(t), \widehat{c}(t)} \Big\{ -\widehat{a}(t)dt + (1-\widehat{b}(t))[\mu^{VC}_t + \widehat{c}(t)^\alpha \widehat{\eta}(t)^\beta - l(t)]dt - \widehat{c}(t)dt \Big\} \tag{82}$$

We can now use the arguments described in detail in Section 4 to show that the optimal controls $(a^*(t), b^*(t), c^*(t))$ are as described in Theorem 1. Further, the optimal termination time is the solution of the optimal stopping problem described in Section 7. The Markov control policy can be shown to satisfy the conditions of the dynamic programming verification theorem (see Section 11 of Oksendal, 2003). Hence, it is, in fact, the optimal control policy among the space of all square-integrable $\{\mathcal{F}_t\}$-adapted controls. This completes the proof of Theorem 1. ▮

Appendix C: Proofs of Remaining Results

The incremental change in termination value (1) depends on η only through the terms η^β, η^γ. There is no loss of generality if the unit of effort is redefined as $z := \eta^\beta$, the production function is taken as $c^\alpha z$ and the disutility of effort is taken as $z^{\gamma/\beta}$. As characterized in Theorem 1, the equilibrium depends on the parameters β and γ only through their ratio γ/β. To simplify the notation in the proofs to follow, we shall hereafter normalize β to 1.

For each parameter "Π" (e.g. σ_0^2, s^2, λ, Δ_0, k) we let $b_t(\pi)$ denote the solution to (35) at time t, define $c_t(\pi) := c(b_t(\pi))$, and let $b(\pi)$ and $c(\pi)$ denote the corresponding time paths when the parameter Π's value equals π. We shall write $F'_t(b, \pi)$ when we wish to make explicit the functional dependence of the derivative of F_t on the parameter value π. For subsequent reference, the derivative of the VC's objective function (34) is given by

$$F'_t(b) = \Delta_t - p + Kc'(b) = \frac{s^2}{s^2 + t\sigma_0^2}\,\Delta_0 - \lambda s + Kc'(b). \tag{83}$$

Recall that a strictly concave differentiable function $f(\cdot)$ possesses a very simple but extremely useful property: the sign of the derivative indicates the direction of the optimum solution, i.e., if $f'(x) > 0$, then $x^* > x$; if $f'(x) < 0$, then $x^* < x$; and if $f'(x) = 0$, then $x^* = x$. The following Lemma will be used repeatedly in the proofs to follow.

Lemma 2

If $F'_t(b, \pi)$ is an increasing (decreasing) function of π, then $b_t(\pi)$ is an increasing (decreasing) function of π.

Proof. Let $\pi^1 < \pi^2$. Suppose first that $F_t'(b, \pi)$ is an increasing function of π. By definition,

$$0 = F_t'(b_t(\pi^2), \pi^2) = F_t'(b_t(\pi^1), \pi^1) < F_t'(b_t(\pi^1), \pi^2),$$

which immediately implies $b_t(\pi^1) < b_t(\pi^2)$ by the strict concavity of F_t. The proof in the decreasing case is analogous. ∎

Proof of Proposition 1. The optimal investment function is clearly positive on $(0, 1]$. The marginal optimal investment is given by

$$c'(b) \quad \propto \quad \left(\frac{1}{k}\right)^{\frac{1}{(1-\alpha)\gamma-1}} b^{r_1}(\gamma - b)^{r_2}(1 - b) \tag{84}$$

where $r_1 := \frac{2-(1-\alpha)\gamma}{(1-\alpha)\gamma-1}$ and $r_2 := \frac{\alpha\gamma}{(1-\alpha)\gamma-1}$ and where the symbol $\propto$ means "equal up to a positive multiplicative constant". Under Assumption 2, the parameter r_2 is positive and the parameter r_1 is negative. (Keep in mind that β is now 1.) Since $\gamma > 1$ (Assumption 1), it follows that $c'(\cdot) > 0$ on $[0, 1)$, $c'(1) = 0$ and $c'(\cdot)$ is negative on $[1, \gamma/\beta)$. This establishes that $c(\cdot)$ is increasing on $[0, 1]$ and achieves it maximum value at $b = 1$. The second derivative is given by

$$c''(b) \quad \propto \quad b^{r_1-1}(\gamma - b)^{r_2-1}[r_1(\gamma - b)(1 - b) - r_2 b(1 - b) - b(\gamma - b)].$$

The function $c(\cdot)$ is strictly concave on $(0, 1]$ since $c''(\cdot)$ is negative on $(0, 1]$. Part (a) has been established. Given that Δ_t decreases with t, it follows directly from Assumption 2 and part (a) that the solution to (35) will belong to $(0, 1)$. On this interval, the VC's objective function $F_t(\cdot)$ is strictly concave by part (a), and hence possesses a unique maximum. This established part (b). ∎

Proof of Theorem 2. It follows directly from (83) and Lemma 2 that the EN's pay performance sensitivity decreases with time where here the parameter $\Pi = t$. This in turn implies the investment rates decrease with time by Proposition 1. Since both the pay performance sensitivities and investment rates decrease with time, the effort levels decrease with time, too—see (20). ∎

Proof of Theorem 3. We begin with parts (a)-(c). By Proposition (1) and the form for the optimal effort level (20), the results for the optimal investment rate and EN's effort paths will follow if we show that the path of the EN's pay performance sensitivity is a pointwise decreasing function of the respective parameters. This follows directly from (83) and Lemma 2 for either σ_0^2 or λ. We turn our attention to s^2. Fix s such that $\Delta_0 < \lambda s$. We have

$$\frac{\partial F_t'(b_t(s), s)}{\partial s} = \frac{2s\Delta_0 t\sigma_0^2}{(s^2 + t\sigma_0^2)^2} - \lambda < \lambda \left[\frac{2t\sigma_0^2 s^2}{(s^2 + t\sigma_0^2)^2} - 1\right] = -\lambda\left[\frac{(s^4 + (t\sigma_0^2)^2}{(s^2 + t\sigma_0^2)^2}\right] < 0.$$

Consequently, $\partial F_t'(b_t(s), s)/\partial s < 0$, which establishes the result for the intrinsic risk by direct application of Lemma 2. The proof of parts (a)-(c) is complete. Part (d) follows directly from

(83) and Lemma 2 and the same arguments given in the proof of parts (a)-(c). As for part (e), an examination of (84) shows that if $c'(b) > 0$, then $c'(b)$ decreases with k. Since $c'(\cdot)$ is positive on $[0, 1)$, the result now follows directly from (83) and Lemma 2 and previous arguments. ▮

We now make explicit the functional dependence of the VC's continuation value (33) on her current assessment of the project's intrinsic quality and write it as $CV_t(\mu_t)$. We drop the superscript on μ_t since it shall always refer to the VC's assessment. Let Z denote the standard normal random variable. We note the continuation value may be expressed as

$$CV_t(\mu_t) = \big[F_t^* + \mu_t - l_t\big]dt + e_t(\mu_t), \tag{85}$$

where

$$e_t(\mu_t) := E_t\Big[\max\{CV_{t+dt}(\mu_{t+dt}), 0\}\Big] = E\Big[\max\{CV_{t+dt}(\mu_t + \sigma_t^{\mu} Z dt), 0\}\Big], \tag{86}$$

where the equality follows from (5).

Proof of Proposition 2. Pick an $\epsilon > 0$. Since the function $e_{T-dt}(\cdot) \equiv 0$, the function $CV_{T-dt}(\cdot)$ is an increasing function of μ_{T-dt}. Moreover, $CV_{T-dt}(\mu_{T-dt} + \epsilon) - CV_{T-dt}(\mu_{T-dt}) = \epsilon dt$. It then follows from (86) that $e_{T-2dt}(\cdot)$ is a nondecreasing function of μ_{t-2dt} and that $e_{T-2dt}(\mu_{t-2dt}+\epsilon) - e_{T-2dt}(\mu_{t-2dt}) \leq \epsilon dt$, too. Consequently, $CV_{T-2dt}(\mu_{T-2dt})$ increases with μ_{T-2dt} and $CV_{T-2dt}(\mu_{T-2dt}+\epsilon) - CV_{T-2dt}(\mu_{T-2dt})$ is bounded above by $2\epsilon dt$. Working backwards through time, the argument can be repeated to show that at each time t the function $CV_t(\cdot)$ increases with μ_t and that $CV_t(\mu_t + \epsilon) - CV_t(\mu_t) \leq \epsilon(T - t)$. The latter property immediately establishes that $CV_t(\cdot)$ is a continuous function of μ_t as ϵ was chosen arbitrarily.

Since the Δ_t decreases with t each $F_t(\cdot)$ is bounded above by $B := F_0(b_0^*)$. Fix time t. Since the expected rate of within period flow at any future point in time is bounded above by $\mu_t + B$, it follows that $e_t(\mu_t)$ is bounded above by $\max\{(\mu_t + B)(T - t), 0\}$. It follows then that $CV_t(\mu_t)$ is negative for sufficiently small μ_t. Since $CV_t(\mu_t)$ is obviously positive for sufficiently high μ_t and $CV_t(\cdot)$ is continuous, there exists a unique value μ_t^* for which $CV_t(\mu_t^*) = 0$. Clearly, the VC should terminate only if $\mu_t < \mu_t^*$. ▮

Proof of Proposition 3. The objective function $F_t(\cdot)$ (34) is an increasing function of Δ_0, which implies that F_t^* is also a increasing function of Δ_0. One may proceed exactly as in the proof of Proposition 2 to establish that each $CV_t(\cdot)$ is a pointwise increasing function of Δ_0. Consequently, the trigger values will decrease. Since a change in Δ_0 has no effect on the sample paths, part (a) follows. The proof of (b) is the same, except that each $F_t(\cdot)$ is now a decreasing function of either λ or k, and thus the trigger values will increase. ▮

Proof of Proposition 4. Pick $\epsilon > 0$ and define θ_0 so that $P(\Theta > \theta_0) = \epsilon$. Now

$$\begin{aligned} P(\tau > T) &= P\{\mu_t > \mu_t^* \text{ for all } t \in [0, T]\} \\ &\leq P\{\mu_T > \mu_T^* \mid \Theta \leq \theta_0\} P(\Theta \leq \theta_0) + P(\Theta > \theta_0) \\ &\leq P\{\mu_T > \mu_T^* \mid \Theta = \theta_0\} + \epsilon. \end{aligned} \tag{87}$$

By Proposition 2 and the assumed property of the l_t, the μ_t^* eventually always lie above a positive constant. Given this fact and the fact that the conditional distribution of $\int_0^t \xi_u du/t$ given $\Theta = \theta_0$ is $N(\theta_0, s^2/t)$,

$$\begin{aligned} P\{\mu_t \geq \mu_t^* \mid \Theta = \theta_0\} &= P\{\frac{s^2\mu_0 + \sigma_0^2(\int_0^t \xi_u du)}{s^2 + t\sigma_0^2} \geq \mu_t^* \mid \Theta = \theta_0\} \\ &\leq P\{\frac{\int_0^t \xi_u du}{t} \geq \mu_t^* + \frac{s^2(\mu_t^* - \mu_0)}{t\sigma_0^2} \mid \Theta = \theta_0\} \to 0 \text{ as } t \to \infty. \end{aligned} \tag{88}$$

The result now follows from (87) and (88) since ϵ was chosen arbitrarily. ▮

Table 1: Baseline values for the parameters

Risk parameters		VC parameters		EN parameters				Technology parameters		
σ_0^2	s^2	R_b	μ_0^{VC}	Δ_0	λ	k	γ	α	β	ℓ_1
0.2202	0.5332	0.15	0.0777	0.3351	0.7890	0.0232	9.3145	0.2034	0.7966	0.0534

Table 2: Predicted and Observed Statistics

	Round-to-Round Expected Returns (in %)				Round-to-Round Standard Deviations (in %)				Expected No. of Rounds
	Rd 1	Rd 2	Rd 3	Rd 4	Rd 1	Rd 2	Rd 3	Rd 4	
Cochrane (2005)	26.0	20.0	15.0	8.8	90	83	77	84	2.10
Model's predictions	25.3	20.2	15.7	10.2	87	85	83	80	2.09

Table 3: Economic Values for Baseline

				Distribution of Project Duration				
Agency Scenario	Firm Value	VC Value	Investment	p_0^*	p_1^*	p_2^*	p_3^*	$E[\tau]$
Actual	11.21	10.02	3.07	0.2981	0.3852	0.2487	0.0637	2.089
Symmetric	9.64	8.77	2.58	0.3339	0.3713	0.2333	.0585	2.026
No Agency	23.14	12.14	7.29	0.1626	0.3702	0.3392	0.1190	2.442

Table 4: Contract Parameter Values (first four periods)

Agency Scenario	b_0^*	b_1^*	b_2^*	b_3^*	c_0^*	c_1^*	c_2^*	c_3^*
Actual	0.195	0.143	0.124	0.114	0.137	0.133	0.131	0.130
Symmetric	0.084	0.084	0.084	0.084	0.125	0.125	0.125	0.125
No Agency	1.000	1.000	1.000	1.000	0.152	0.152	0.152	0.152

Table 5: Comparative Statics

			Firm Value						VC Value						Investment		
		$E[\tau]$	Actual	Symmetric	No Agency	Agency Cost	Actual Cost	Actual Value	Actual	Symmetric	No Agency	Agency Cost	Actual Cost	Actual Value	Actual	Symmetric	No Agency
α	0.10	2.55	23.22	19.29	54.98	0.65	0.58	0.11	20.05	17.02	28.33	0.40	0.29	0.27	3.92	3.19	10.15
	0.15	2.30	15.22	12.91	34.14	0.62	0.55	0.11	13.38	11.58	17.67	0.34	0.24	0.30	3.50	2.88	8.68
	0.20	2.09	11.21	9.64	23.14	0.58	0.52	0.12	10.02	8.77	12.14	0.28	0.17	0.37	3.07	2.58	7.29
	0.25	1.92	8.62	7.50	16.72	0.55	0.48	0.12	7.82	6.91	8.92	0.23	0.12	0.45	2.70	2.27	6.13
	0.30	1.78	6.87	6.05	13.03	0.54	0.47	0.12	6.31	5.64	6.89	0.18	0.08	0.54	2.36	2.01	5.28
	0.35	1.66	5.65	5.03	10.35	0.51	0.45	0.12	5.25	4.73	5.52	0.14	0.05	0.66	2.10	1.80	4.58
	0.40	1.57	4.79	4.31	8.42	0.49	0.43	0.12	4.49	4.09	4.54	0.10	0.01	0.91	1.85	1.60	3.96
	0.45	1.49	4.12	3.75	6.96	0.46	0.41	0.12	3.90	3.59	3.83	0.06	-0.02	1.30	1.67	1.46	3.49
σ_0^2	0.12	2.06	3.79	3.27	6.44	0.49	0.41	0.16	3.36	2.98	3.71	0.20	0.10	0.51	0.99	0.81	1.94
	0.17	2.07	6.07	5.23	11.57	0.55	0.48	0.13	5.40	4.76	6.17	0.23	0.12	0.45	1.61	1.33	3.49
	0.22	2.09	11.21	9.64	23.14	0.58	0.52	0.12	10.02	8.77	12.14	0.28	0.17	0.37	3.07	2.58	7.29
	0.27	2.11	23.11	20.00	49.93	0.60	0.54	0.10	20.83	18.28	27.91	0.35	0.25	0.26	6.28	5.31	18.08
	0.32	2.14	46.50	42.93	125.93	0.66	0.63	0.04	42.03	39.38	74.34	0.47	0.43	0.08	17.22	13.04	50.48
s^2	0.33	2.18	13.99	11.73	26.63	0.56	0.47	0.15	12.31	10.57	13.91	0.24	0.12	0.52	3.64	2.97	7.60
	0.38	2.16	13.03	11.03	25.61	0.57	0.49	0.14	11.52	9.98	13.41	0.26	0.14	0.45	3.45	2.83	7.55
	0.43	2.13	12.27	10.42	24.88	0.58	0.51	0.13	10.90	9.44	12.93	0.27	0.16	0.42	3.30	2.74	7.44
	0.48	2.11	11.71	10.03	23.91	0.58	0.51	0.12	10.43	9.11	12.52	0.27	0.17	0.39	3.17	2.64	7.36
	0.53	2.09	11.21	9.64	23.14	0.58	0.52	0.12	10.02	8.77	12.14	0.28	0.17	0.37	3.07	2.58	7.29
	0.58	2.07	10.75	9.28	22.15	0.58	0.51	0.11	9.64	8.46	11.79	0.28	0.18	0.35	2.94	2.48	7.20
	0.63	2.05	10.34	8.97	21.65	0.59	0.52	0.11	9.29	8.19	11.48	0.29	0.19	0.33	2.88	2.42	7.11
	0.68	2.03	9.91	8.62	21.30	0.60	0.53	0.10	8.92	7.88	11.19	0.30	0.20	0.31	2.78	2.35	7.01
	0.73	2.01	9.60	8.36	20.95	0.60	0.54	0.10	8.67	7.66	10.93	0.30	0.21	0.31	2.61	2.23	6.98
Δ_0	0.19	2.06	10.37	9.64	23.14	0.58	0.55	0.05	9.36	8.77	12.14	0.28	0.23	0.18	2.81	2.58	7.29
	0.24	2.07	10.62	9.64	23.14	0.58	0.54	0.07	9.55	8.77	12.14	0.28	0.21	0.23	2.89	2.58	7.29
	0.29	2.08	10.90	9.64	23.14	0.58	0.53	0.09	9.77	8.77	12.14	0.28	0.19	0.30	2.97	2.58	7.29
	0.34	2.09	11.21	9.64	23.14	0.58	0.52	0.12	10.02	8.77	12.14	0.28	0.17	0.37	3.07	2.58	7.29
	0.39	2.10	11.58	9.64	23.14	0.58	0.50	0.14	10.30	8.77	12.14	0.28	0.15	0.45	3.18	2.58	7.29
	0.44	2.11	12.01	9.64	23.14	0.58	0.48	0.18	10.62	8.77	12.14	0.28	0.13	0.55	3.31	2.58	7.29
	0.49	2.13	12.54	9.64	23.14	0.58	0.46	0.21	11.00	8.77	12.14	0.28	0.09	0.66	3.49	2.58	7.29

Table 6: Dynamics (Asymmetric Case)

		b_0^*	b_1^*	b_2^*	b_3^*	c_0^*	c_1^*	c_2^*	c_3^*	p_0^*	p_1^*	p_2^*	p_3^*	I_0^*	I_1^*	I_2^*	I_3^*
α	0.10	0.25	0.19	0.16	0.15	0.09	0.08	0.08	0.08	0.12	0.36	0.37	0.14	0.09	0.23	0.91	6.61
	0.15	0.22	0.16	0.14	0.13	0.11	0.11	0.11	0.11	0.21	0.39	0.30	0.09	0.11	0.30	1.26	9.67
	0.20	0.20	0.14	0.12	0.11	0.14	0.13	0.13	0.13	0.30	0.39	0.25	0.06	0.14	0.37	1.60	12.79
	0.25	0.17	0.12	0.11	0.10	0.16	0.15	0.15	0.15	0.38	0.36	0.20	0.05	0.16	0.44	1.94	15.89
	0.30	0.15	0.11	0.09	0.09	0.17	0.17	0.16	0.16	0.46	0.33	0.17	0.03	0.17	0.50	2.26	19.15
	0.35	0.13	0.09	0.08	0.07	0.19	0.18	0.18	0.17	0.53	0.30	0.14	0.03	0.19	0.56	2.58	22.15
	0.40	0.12	0.08	0.07	0.06	0.20	0.19	0.19	0.18	0.59	0.27	0.12	0.02	0.20	0.62	2.87	25.01
	0.45	0.10	0.07	0.06	0.06	0.21	0.20	0.19	0.19	0.64	0.24	0.10	0.02	0.21	0.67	3.14	27.62
σ_0^2	0.12	0.20	0.16	0.14	0.13	0.14	0.13	0.13	0.13	0.18	0.58	0.23	0.01	0.14	0.32	1.53	21.88
	0.17	0.20	0.15	0.13	0.12	0.14	0.13	0.13	0.13	0.25	0.46	0.25	0.04	0.14	0.35	1.53	14.47
	0.22	0.20	0.14	0.12	0.11	0.14	0.13	0.13	0.13	0.30	0.39	0.25	0.06	0.14	0.37	1.60	12.79
	0.27	0.20	0.14	0.12	0.11	0.14	0.13	0.13	0.13	0.33	0.34	0.24	0.09	0.14	0.39	1.72	12.87
	0.32	0.20	0.13	0.11	0.11	0.14	0.13	0.13	0.13	0.35	0.30	0.23	0.10	0.14	0.41	1.86	13.87
s^2	0.33	0.34	0.19	0.15	0.14	0.14	0.14	0.13	0.13	0.29	0.35	0.27	0.09	0.14	0.38	1.51	9.84
	0.38	0.29	0.17	0.15	0.13	0.14	0.14	0.13	0.13	0.29	0.36	0.26	0.08	0.14	0.38	1.52	10.45
	0.43	0.25	0.16	0.14	0.13	0.14	0.13	0.13	0.13	0.29	0.37	0.26	0.08	0.14	0.38	1.55	11.16
	0.48	0.22	0.15	0.13	0.12	0.14	0.13	0.13	0.13	0.30	0.38	0.25	0.07	0.14	0.37	1.57	11.93
	0.53	0.20	0.14	0.12	0.11	0.14	0.13	0.13	0.13	0.30	0.39	0.25	0.06	0.14	0.37	1.60	12.79
	0.58	0.18	0.14	0.12	0.11	0.14	0.13	0.13	0.13	0.30	0.39	0.24	0.06	0.14	0.37	1.64	13.76
	0.63	0.16	0.13	0.11	0.11	0.13	0.13	0.13	0.13	0.31	0.40	0.24	0.05	0.13	0.37	1.67	14.82
	0.68	0.15	0.12	0.11	0.10	0.13	0.13	0.13	0.13	0.31	0.41	0.23	0.05	0.13	0.37	1.71	16.01
	0.73	0.14	0.12	0.10	0.10	0.13	0.13	0.13	0.13	0.31	0.41	0.23	0.04	0.13	0.37	1.76	17.34
Δ_0	0.19	0.12	0.11	0.10	0.10	0.13	0.13	0.13	0.13	0.32	0.38	0.24	0.06	0.13	0.35	1.53	12.17
	0.24	0.14	0.12	0.11	0.10	0.13	0.13	0.13	0.13	0.31	0.38	0.24	0.06	0.13	0.36	1.55	12.35
	0.29	0.16	0.13	0.12	0.11	0.13	0.13	0.13	0.13	0.30	0.38	0.25	0.06	0.13	0.36	1.58	12.55
	0.34	0.20	0.14	0.12	0.11	0.14	0.13	0.13	0.13	0.30	0.39	0.25	0.06	0.14	0.37	1.60	12.79
	0.39	0.24	0.16	0.13	0.12	0.14	0.13	0.13	0.13	0.29	0.39	0.25	0.06	0.14	0.38	1.64	13.09
	0.44	0.30	0.18	0.14	0.13	0.14	0.14	0.13	0.13	0.28	0.39	0.26	0.07	0.14	0.39	1.68	13.43
	0.49	0.41	0.20	0.16	0.13	0.15	0.14	0.13	0.13	0.27	0.40	0.26	0.07	0.15	0.40	1.73	13.87

Table 7: Implied Discount Rates (in %)

		Firm Value	VC Value
α	0.10	48.5	48.5
	0.15	49.5	49.5
	0.20	49.5	49.5
	0.25	49.5	49.5
	0.30	50.0	50.0
	0.35	50.0	50.0
	0.40	50.5	50.5
	0.45	51.0	51.0
σ_0^2	0.12	51.0	50.5
	0.17	50.5	50.5
	0.22	49.5	49.5
	0.27	48.5	48.5
	0.32	51.0	51.0
s^2	0.33	47.5	47.5
	0.38	48.5	48.5
	0.43	49.0	49.0
	0.48	49.5	49.5
	0.53	49.5	49.5
	0.58	50.0	50.0
	0.63	50.0	50.0
	0.68	50.5	50.5
	0.73	50.5	50.5
Δ_0	0.19	34.0	34.0
	0.24	39.5	39.5
	0.29	44.5	44.5
	0.34	49.5	49.5
	0.39	54.5	54.5
	0.44	59.5	59.5
	0.49	64.5	64.5

Table 8: Implied Discount Rates (in %)

		Firm Value					VC Value				
		σ_0^2					σ_0^2				
		0.12	0.17	0.22	0.27	0.32	0.12	0.17	0.22	0.27	0.32
s^2	0.33	49.0	48.5	47.5	47.5	47.5	48.5	48.0	47.5	47.5	47.5
	0.38	49.5	49.0	48.5	47.5	48.5	49.0	48.5	48.5	47.5	48.5
	0.43	50.0	49.5	49.0	47.5	48.5	49.5	49.0	49.0	47.5	48.5
	0.48	50.5	50.5	49.5	47.5	51.0	50.0	50.0	49.5	47.5	51.0
	0.53	51.0	51.0	50.0	48.5	51.0	50.5	50.5	50.0	48.5	51.0
	0.58	51.5	51.5	50.0	49.0	50.0	51.0	51.0	50.0	49.0	50.0
	0.63	51.5	52.0	50.5	49.0	49.0	51.0	51.5	50.5	49.0	49.0
	0.68	52.0	52.5	51.0	49.5	50.0	51.5	52.5	51.0	49.5	50.0
	0.73	52.0	52.0	51.0	50.0	49.5	51.5	52.0	50.5	50.0	50.0