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Publisher Taylor & Francis

Informa Ltd Registered in England and Wales Registered Number: 1072954 Registered office: Mortimer House, 37-41 Mortimer Street, London W1T 3JH, UK



## Scandinavian Actuarial Journal

Publication details, including instructions for authors and subscription information:

<http://www.informaworld.com/smpp/title-content=t713690025>

### On the ordering of ruin probabilities for the surplus process perturbed by diffusion

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First Published on: 01 May 2008

**To cite this Article** Chi-Liang Tsai, Cary(2008)'On the ordering of ruin probabilities for the surplus process perturbed by diffusion',Scandinavian Actuarial Journal,99999:1,

**To link to this Article:** DOI: 10.1080/03461230801980417

**URL:** <http://dx.doi.org/10.1080/03461230801980417>

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## Original Article

# On the ordering of ruin probabilities for the surplus process perturbed by diffusion

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(Accepted 11 February 2008)

In this paper, we study orders of pairs of ruin probabilities resulting from two individual claim size random variables for corresponding continuous time surplus processes perturbed by diffusion with different premium rates, relative security loadings, and variance parameters of the diffusion processes. We show that high frequency and low severity risks yield smaller ruin probabilities than low frequency and high severity risks. These ordering relationships can also be used to obtain upper and/or lower bounds on ruin probabilities. Finally, some examples are given to illustrate the results of the theorems.

*Keywords:* Surplus process; Diffusion process; Ruin probability; Maximal aggregate loss; Compound geometric distribution; Ordering; Bound

## 1. Introduction

Consider the classical continuous time surplus process at time  $t$ ,

$$U(t) = u + ct - S(t), \quad t \geq 0, \quad (1.1)$$

where  $u = U(0)$  is the initial surplus,  $c$  is the constant rate per unit time at which the premiums are received, and  $S(t) = X_1 + X_2 + \dots + X_{N(t)}$  is the aggregate claims up to time  $t$ . The number of claims,  $N(t)$ , is assumed to follow a Poisson process with parameter  $\lambda$ . The individual claim sizes  $X_1, X_2, \dots$ , independent of  $N(t)$ , are positive, independent and identically distributed (as  $X$ ) random variables with common distribution function  $F(x) = \Pr(X \leq x)$  and moments  $\mu_n = E[X^n] = \int_0^\infty x^n dF(x)$  for  $n = 0, 1, 2, \dots$ . We call  $\{S(t); t \geq 0\}$  ( $S(t) = 0$  if  $N(t) = 0$ ) a compound Poisson process with parameter  $\lambda$ , and we assume  $c = \lambda\mu_1(1 + \theta)$ , where  $\theta > 0$  is the relative security loading.

Let  $T = \inf\{t: U(t) < 0\}$  be the time of ruin (the first time that the surplus becomes negative), and  $\psi(u) = \Pr(T < \infty | U(0) = u)$  be the probability of ruin for the surplus process (1.1). The deficit at the time of ruin,  $|U(T)|$ , and the surplus immediately before the time of ruin,  $U(T-)$  ( $U(T-)$  is the left limit of  $U(t)$  at  $t = T$ ), are two important non-negative random variables in connection with the time of ruin  $T$ .

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An independent diffusion (or Wiener) process was added to Eq. (1.1) by Gerber (1970) by extending the classical surplus process so that

$$U(t) = u + ct - S(t) + \sigma W(t), \quad t \geq 0, \quad (1.2)$$

where  $\sigma > 0$  ( $\sigma^2$  is called the variance parameter) and  $\{W(t): t \geq 0\}$  is a standard Wiener process that is independent of the compound Poisson process  $\{S(t): t \geq 0\}$  and of the individual claim sizes  $X_1, X_2, \dots$ . In this case, the definition of the time of ruin is modified as  $T = \inf\{t: U(t) \leq 0\}$  ( $T = \infty$  if the set is empty) due to the oscillation character of the added Wiener process. Dufresne & Gerber (1991) studied three kinds of probabilities based on model (1.2):  $\psi_d(u)$ , the probability of ruin caused by oscillation;  $\psi_s(u)$ , the probability of ruin caused by a claim; and  $\psi_t(u)$ , the probability of ruin. That is,  $\psi_d(u) = \Pr(T < \infty, U(T) = 0 | U(0) = u)$ ,  $\psi_s(u) = \Pr(T < \infty, U(T) < 0 | U(0) = u)$ , and  $\psi_t(u) = \psi_d(u) + \psi_s(u) = \Pr(T < \infty | U(0) = u)$ ,  $u \geq 0$ .

Next, let  $G(y) = H * \Gamma(y) = \int_0^y H(y-t) d\Gamma(t)$  where  $\Gamma(y) = \int_0^y \bar{F}(t) dt / E[X]$ ,  $H(y) = 1 - e^{-(c/D)y}$ ,  $\bar{F}(y) = 1 - F(y)$  and  $D = \sigma^2/2$ . Tsai (2003) showed that  $\bar{K}(u)$  satisfies the defective renewal equation

$$\bar{K}(u) = \frac{1}{1+\theta} \int_0^u \bar{K}(u-x) dG(x) + \frac{1}{1+\theta} \bar{G}(u), \quad u \geq 0, \quad (1.3)$$

and

$$\bar{K}(u) = \frac{1}{1+\theta} \psi_d(u) + \psi_s(u) \sum_{n=1}^{\infty} \frac{\theta}{1+\theta} \left[ \frac{1}{1+\theta} \right]^n \bar{G}^{*n}(u), \quad u \geq 0, \quad (1.4)$$

is a compound geometric tail distribution with parameter  $1/\theta$  (Klugman *et al.* (2004), p. 240) and  $\bar{K}(0) = 1/(1+\theta)$ . Moreover, he proved that

$$\psi_t(u) = \bar{K} * \bar{H}(u) = \sum_{n=0}^{\infty} \frac{\theta}{1+\theta} \left[ \frac{1}{1+\theta} \right]^n \bar{G}^{*n} * \bar{H}(u), \quad u \geq 0, \quad (1.5)$$

is also the tail of a compound geometric convolution. Equation (1.5) can also be obtained from a probabilistic viewpoint. Consider the aggregate loss at time  $t$ ,  $L(t) = u - U(t) = S(t) - ct - \sigma W(t)$ , and the maximal aggregate loss,  $L = \max\{L(t): t \geq 0\}$ . Let  $T_n$  be the time when the  $n$ -th record high of the aggregate loss process  $\{L(t): t \geq 0\}$  (or the  $n$ -th record low of the process  $\{U(t): t \geq 0\}$ ) is caused by a claim (we set  $T_0 = 0$ ). Define

$$L_{o,n} = \max\{L(t): t < T_{n+1}\} - L(T_n) = U(T_n) - \min\{U(t): t < T_{n+1}\}, \quad n = 0, 1, \dots, N,$$

and

$$L_{c,n} = L(T_n) - L(T_{n-1}) - L_{o,n-1} = U(T_{n-1}) - U(T_n) - L_{o,n-1}, \quad n = 1, 2, \dots, N,$$

as the amount that result in the  $(n+1)$ -th and  $n$ -th record highs of the aggregate loss process  $\{L(t)\}$  due to oscillation and a claim, respectively, where  $N$  is the number of record highs of the process  $\{L(t)\}$  caused by a claim. Note that  $L_{c,n} > 0$  and  $L_{o,n} \geq 0$  ( $L_{o,n}$  could be zero for some  $n$  because of two consecutive  $L_{c,n}$  and  $L_{c,n+1}$ ). Then, the random variable  $L$  can be decomposed (Dufresne & Gerber (1991)) as

$$L = L_{o,0} + L_{c,1} + L_{o,1} + \dots + L_{c,N} + L_{o,N} = \sum_{n=1}^N (L_{o,n-1} + L_{c,n}) + L_{o,N} \quad (1.6)$$

(we set  $T_{N+1} = \infty$ ) with  $L = L_{o,0}$  if  $N=0$  (see Figure 1). Then  $g(y)dy$  (or equivalently,  $[1/(1 + \theta)]dG(y)$ , where  $G(y) = \int_0^y g(x)dx / \int_0^\infty g(x)dx = (1 + \theta) \int_0^y g(x)dx$ ) is the probability (or the probability  $(\lambda/c)\bar{F}(y)dy$  for the case  $D=0$ ) that the first record low of  $\{U(t)\}$  caused by a claim is between  $u-y$  and  $u-y-dy$  (see also Gerber & Landry (1998)) where  $g(y) = (\lambda/D) \int_0^y e^{-(c/D)(y-s)} \bar{F}(s) ds$ . Because a compound Poisson process has stationary and independent increments,  $\int_0^\infty g(y)dy = 1/(1 + \theta)$  is the probability that there is a record low of  $\{U(t)\}$  (or a record high of  $\{L(t)\}$ ) caused by a claim, and  $\theta/(1 + \theta)$  is the probability that there is no record high of  $\{L(t)\}$  caused by a claim. Then  $N$  has a geometric distribution function with parameter  $1/\theta$ , that is,  $\Pr(N = n) = [\theta/(1 + \theta)] \times [1/(1 + \theta)]^n$ ,  $n = 0, 1, 2, \dots$ . In addition, the random variables  $L_{o,0}, L_{o,1}, L_{o,2}, \dots$  are identically distributed (as  $L_o$ ) with common distribution function  $F_o$ , and  $L_{c,1}, L_{c,2}, L_{c,3}, \dots$  are identically distributed (as  $L_c$ ) with common distribution function  $F_c$ ; also,  $N, L_{o,0}, L_{c,1}, L_{o,1}, L_{c,2}, L_{o,2}, \dots$  are independent; then

$$\Pr(L > u) = \sum_{n=0}^\infty \Pr(L > u | N = n) \Pr(N = n) \sum_{n=0}^\infty \left(\frac{\theta}{1 + \theta}\right) \left(\frac{1}{1 + \theta}\right)^n \overline{F_o^{*(n+1)} * F_c^{*(n)}}(u).$$

Thus,  $\psi_t(u)$  can be treated as  $\psi_t(u) = \Pr(L > u)$  (the tail probability of  $L$ ) with  $\psi_t(\infty) = 0$ ,  $F_c = \Gamma$ ,  $F_o = H$  and  $F_c * F_o = \Gamma * H = G$  (Dufresne & Gerber (1991)), which gives a probabilistic interpretation for the expression for  $\psi_t(u)$  in Eq. (1.5). Similarly, the expression for  $\bar{K}(u)$  in Eq. (1.4) can be considered as  $\bar{K}(u) = \Pr(L^* > u)$  (the tail probability of  $L^*$ ) where

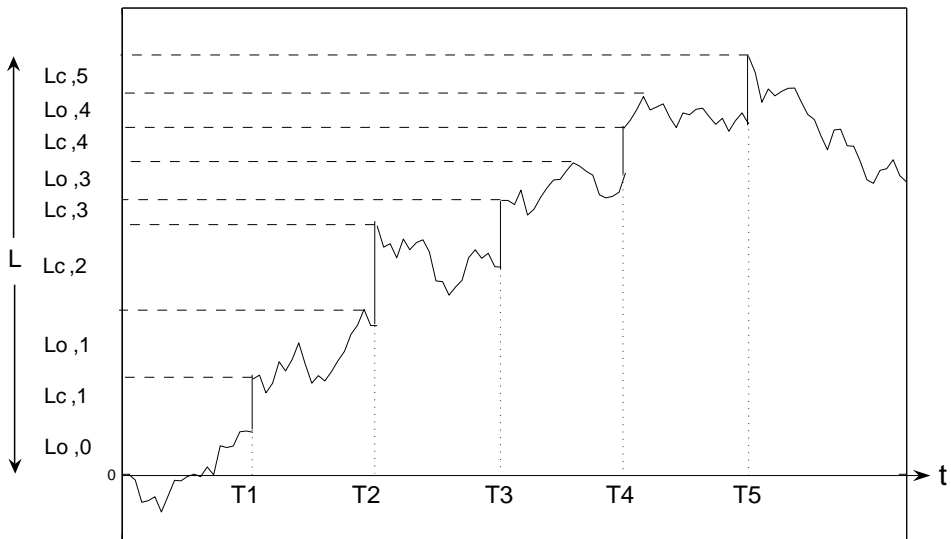


Figure 1. Interpretation of  $L_{o,s}$  and  $L_{c,s}$ .

$$L^* = L_{o,0} + L_{c,1} + \cdots + L_{o,N-1} + L_{c,N} = \sum_{n=1}^N (L_{o,n-1} + L_{c,n}) \quad (1.7)$$

with  $L^* = 0$  if  $N = 0$ . Moreover, when the diffusion component is removed from Eq. (1.2), that is,  $\sigma = 0$ , then all these  $L_{o,s}$  disappear which implies both  $L$  and  $L^*$  reduce to  $L^* = \sum_{n=1}^N L_{c,n}$ , and the probability of ruin for Eq. (1.1) becomes (Klugman *et al.* (2004))

$$\psi(u) = \Pr(L^* > u) = \sum_{n=1}^{\infty} \frac{\theta}{1 + \theta} \left[ \frac{1}{1 + \theta} \right]^n \Gamma^{\overline{m}}(u), \quad u \geq 0, \quad (1.8)$$

a compound geometric tail distribution with parameter  $1/\theta$  and  $\psi(0) = 1/(1 + \theta)$ .

Orders have been studied in the actuarial literature, and broadly used in casualty and health insurance to compare underlying risks (see Goovaerts *et al.* (1984, 1990), Kaas *et al.* (1994), Kaas & Hesselager (1995), Cheng & Pai (2003), and Tsai (2006)). This paper studies orders between pairs of ruin probabilities based on two surplus processes perturbed by diffusion. The remainder of the paper is organized as follows. Section 2 gives conditions and theorems for obtaining ordering relationships for  $\overline{K}$ ,  $\psi_t$  and  $\psi_d$  resulting from claim size random variables  $X$  and  $Y$  for corresponding continuous time surplus processes perturbed by diffusion, respectively. In Section 3, we apply the theorems proposed in Section 2 to random variables distributed as a single Exponential, a mixture of Exponentials, Gamma, Pareto and Lognormal, respectively. The ordering relationships can be used to obtain upper and/or lower bounds on  $\overline{K}$  and  $\psi_t$ . Corresponding examples are also given to illustrate the results of the theorems. In the appendix, we propose definitions, lemmas and examples for a variety of orders between two random variables used in this paper.

## 2. Ordering of ruin probabilities

In this section, we give conditions and theorems for the orders of pair  $\overline{K}_X$  and  $\overline{K}_Y$ , pair  $\psi_{t,X}$  and  $\psi_{t,Y}$ , and pair  $\psi_{d,X}$  and  $\psi_{d,Y}$ , all resulting from the individual claim size random variables  $X$  and  $Y$  for two continuous time surplus processes perturbed by diffusion with different premium rates, relative security loadings, and variance parameters of the diffusion processes. We show that these conditions are more flexible than existing ones, and thus provide more applications. Some important remarks are also proposed.

Let  $Z = X, Y$ , consider the following surplus processes:

$$U_Z(t) = u + c_Z t - S_Z(t) + \sigma_Z W(t), \quad t \geq 0, \quad (2.1)$$

where  $S_Z(t) = \sum_{i=1}^{N_Z(t)} Z_i$ . We suppose that the number of claims,  $N_Z(t)$ , has the parameters  $\lambda_Z$ , the claim size random variables  $Z_1, Z_2, \dots$  are i.i.d. and distributed as  $Z$  with d.f.  $F_Z$ ,  $D_Z = \sigma_Z^2/2$ , and the premium rate  $c_Z = \lambda_Z E(Z)(1 + \theta_Z)$ . The i.i.d. random variables which correspond to  $\{L_{o,k}: k = 0, 1, \dots\}$  and  $\{L_{c,k}: k = 1, 2, \dots\}$  are  $\{L_{o,k}^Z: k = 0, 1, \dots\}$  (distributed as  $L_o^Z$  with d.f.  $H_Z$ ) and  $\{L_{c,k}^Z: k = 1, 2, \dots\}$  (distributed as  $L_c^Z$  with d.f.  $\Gamma_Z$ ) for the surplus process  $U_Z$ . Moreover, let  $\overline{K}_Z(u) = \Pr[L_Z^* > u]$  be the corresponding tail probability of  $L_Z^*$

(see Eq. (1.7) for the definition of  $L^*$ ),  $\psi_{t,Z}(u) = \Pr[L_Z > u]$  be the corresponding tail probability of  $L_Z$  (see Eq. (1.6) for the definition of  $L$ ), and  $N_Z$  be the corresponding number of record highs  $N$  of  $\{L(t)\}$  for the surplus processes  $U_Z$ . Finally, let  $\psi_Z(u) = \Pr[L_Z^\bullet > u]$  be the corresponding tail probability of  $L_Z^\bullet$  (see Eq. (1.8) for the definition of  $L^\bullet$ ) for the case that the diffusion component is removed.

First, let us start from a special case that  $c_X = c_Y$ ,  $D_X = D_Y$  and  $\lambda_X = \lambda_Y$ . Suppose  $E(X) = E(Y)$  which implies  $\theta_X = \theta_Y$ ; if  $X \leq_{sl} Y$  then  $\psi_X(u) \leq \psi_Y(u)$  for the surplus process (1.1) (see Goovaerts *et al.* (1984, 1990)). Cheng & Pai (1999) extended the result thus – if  $X \leq_{sl(n)} Y$  then  $\psi_X(u) \leq_{sl(n-1)} \psi_Y(u)$ . Tsai (2006) further generalized the ordering of ruin probabilities with the following theorem based on model (1.2):

**THEOREM 1.** *Let  $c_X = c_Y$ ,  $D_X = D_Y$  and  $\lambda_X = \lambda_Y$ . Suppose  $E(X) = E(Y)$ ; then  $X \leq_{sl(n)} Y \Rightarrow \bar{K}_X \leq_{sl(n-1)} \bar{K}_Y$ ,  $\psi_{t,X} \leq_{sl(n-1)} \psi_{t,Y}$  and  $\psi_{d,X} \leq_{sl(n)} \psi_{d,Y}$ .*

Let the mean time to ruin due to oscillation be denoted by  $\psi_{d;1}(u) = E[TI(T < \infty, U(T) = 0 | U(0) = u)]$ . Then we have the following result for the  $n$ -th stop-loss order between  $\psi_{d;1,X}(u)$  and  $\psi_{d;1,Y}(u)$ .

**THEOREM 2.** *Let  $c_X = c_Y$ ,  $D_X = D_Y$  and  $\lambda_X = \lambda_Y$ . Suppose  $E(X) = E(Y)$ ; then  $X \leq_{sl(n)} Y \Rightarrow \psi_{d;1,X} \leq_{sl(n)} \psi_{d;1,Y}$ .*

*Proof.* From Theorem 1, we have  $\psi_{t,X} \leq_{sl(n-1)} \psi_{t,Y}$  and  $\psi_{d,X} \leq_{sl(n)} \psi_{d,Y}$ . Since the low degree stop-loss order implies high degree stop-loss order (Goovaerts *et al.* (1990)), we have  $\psi_{t,X} \leq_{sl(n)} \psi_{t,Y}$ . Corollary 7 of Tsai & Willmot (2002) showed that  $\psi_{d;1}$  is directly proportional to convolution to  $\psi_t$  and  $\psi_d$ , that is,

$$\psi_{d;1}(u) = \frac{1 + \theta}{\lambda\mu_1\theta^2} \int_0^u \psi_t(u-x) \frac{\theta}{1 + \theta} \psi_d(x) dx = \frac{1 + \theta}{c\theta} \int_0^u \psi_t(u-t) \psi_d(t) dt. \quad (2.2)$$

Because the  $n$ -th stop-loss order is preserved under addition (Tsai (2006)), we have  $\psi_{d;1,X} \leq_{sl(n)} \psi_{d;1,Y}$ . □

**COROLLARY 1.** *Let  $c_X = c_Y$ ,  $D_X = D_Y$  and  $\lambda_X = \lambda_Y$ . Suppose  $E(X) = E(Y)$ ; then  $X \leq_{sl} Y \Rightarrow \bar{K}_X \leq_{sl} \bar{K}_Y$ ,  $\psi_{t,X} \leq_{sl} \psi_{t,Y}$ ,  $\psi_{d,X} \leq_{sl} \psi_{d,Y}$  and  $\psi_{d;1,X} \leq_{sl} \psi_{d;1,Y}$ .*

As mentioned in Tsai (2006),  $\bar{K}_X \leq_{sl} \bar{K}_Y$  and  $\psi_{t,X} \leq_{sl} \psi_{t,Y}$  are equivalent to  $\bar{K}_X(u) \leq \bar{K}_Y(u)$  and  $\psi_{t,X}(u) \leq \psi_{t,Y}(u)$ , respectively, for all  $u \geq 0$ .

The constraints  $\lambda_X = \lambda_Y$  and  $E(X) = E(Y)$  limit applications in comparing ruin probabilities based on different expected frequencies of claims and expected severities of claims. Let  $c_X = c_Y$ ,  $D_X = D_Y$ ,  $\lambda_X \geq \lambda_Y$  and  $E(X) \leq E(Y)$  with  $\lambda_X E(X) = \lambda_Y E(Y)$  (that is,  $E[S_X(1)] = E[S_Y(1)]$ ); we wonder which of high frequency and low severity risk  $X$  or low frequency and high severity risk  $Y$  (both have equal expected aggregate claims per unit time) produces smaller ruin probability. The answer for ruin probability resulting from exponential individual claim sizes based on model (1.1) is obvious. Recall that if the claim

amount random variables have common d.f.  $F(t) = 1 - e^{-\beta t}$  then  $\psi(u) = [1/(1 + \theta)]e^{-\theta\beta u/(1+\theta)} = [1/(1 + \theta)]e^{-\theta u(\lambda/c)}$ ; when  $c$  and  $\theta$  are fixed, high frequency and low severity risk yields smaller ruin probability than low frequency and high severity risk, which is consistent with our intuition. For other individual claim size distributions, however, we generally cannot find analytical expressions for  $\psi_t(u)$ ; therefore, it is difficult to give theoretical support for our intuition. To study this problem more generally, we propose the following conditions:

- **Condition 1:**  $X \leq_{mrl} Y$ , or  $e_X(t) \leq e_Y(t)$  for all  $t \geq 0$ .
- **Condition 2:**  $c_X/D_X \geq c_Y/D_Y$ .
- **Condition 3:**  $\theta_X \geq \theta_Y$ , or  $\theta_X/(1 + \theta_X) \geq \theta_Y/(1 + \theta_Y)$ .
- **Condition 4:**  $[c_X/D_X][\theta_X/(1 + \theta_X)] \geq [c_Y/D_Y][\theta_Y/(1 + \theta_Y)]$ .

**LEMMA 1.** *If Condition 1 holds then  $L_c^X \leq_{st} L_c^Y$ , or equivalently,  $\bar{\Gamma}_X(u) \leq \bar{\Gamma}_Y(u)$  for all  $u \geq 0$ .*

*Proof.* Since

$$\frac{1}{e_Z(t)} = \frac{\bar{F}_Z(t)}{\int_t^\infty \bar{F}_Z(u)du} = -\frac{\left(\int_t^\infty \bar{F}_Z(u)du\right)'}{\int_t^\infty \bar{F}_Z(u)du} = -\frac{d}{dt} \log\left(\int_t^\infty \bar{F}_Z(u)du\right) \Rightarrow \int_t^\infty \bar{F}_Z(u)du = ke^{-\int_0^t [1/e_Z(s)]ds}$$

where  $k = \int_0^\infty \bar{F}_Z(u)du = E(Z)$  and  $Z = X, Y$ , we have

$$\frac{\bar{\Gamma}_X(u)}{\bar{\Gamma}_Y(u)} = \frac{\int_u^\infty \bar{F}_X(s)ds/E(X)}{\int_u^\infty \bar{F}_Y(s)ds/E(Y)} = e^{-\int_0^u [1/e_X(s) - 1/e_Y(s)]ds} \leq 1$$

for all  $u \geq 0$ . Because  $L_c^Z$  has the distribution function  $\Gamma_Z$  ( $Z = X, Y$ ), we conclude that  $L_c^X \leq_{st} L_c^Y$  by Lemma 9.  $\square$

**LEMMA 2.** *If Condition 2 holds then  $L_o^X \leq_{st} L_o^Y$ , or equivalently,  $\bar{H}_X(u) \leq \bar{H}_Y(u)$  for all  $u \geq 0$ .*

*Proof.*  $c_X/D_X \geq c_Y/D_Y \Rightarrow \bar{H}_X(u) = e^{-(c_X/D_X)u} \leq e^{-(c_Y/D_Y)u} = \bar{H}_Y(u) \Rightarrow L_o^X \leq_{st} L_o^Y$  since  $L_o^Z$  has the distribution function  $H_Z$  ( $Z = X, Y$ ).  $\square$

**LEMMA 3.** *If Conditions 1 and 2 hold then  $L_c^X + L_o^X \leq_{st} L_c^Y + L_o^Y$ , or equivalently,  $\bar{G}_X(u) \leq \bar{G}_Y(u)$  for all  $u \geq 0$ .*

*Proof.* Since stochastic order is preserved under addition, and  $L_c^Z + L_o^Z$  has the distribution function  $G_Z$  ( $Z = X, Y$ ), the conclusion is easily reached.  $\square$

**LEMMA 4.** *If Condition 3 holds then  $N_X \leq_{lr} N_Y$ , which implies  $N_X \leq_{st} N_Y$ .*

*Proof.* Since  $N_Z$  has the geometric distribution function with parameter  $1/\theta_Z$  ( $Z=X, Y$ ) and  $1/\theta_X \leq 1/\theta_Y$ , by Example 6 in the Appendix we have  $N_X \leq_{lr} N_Y$ , hence  $N_X \leq_{st} N_Y$ .  $\square$

**THEOREM 3.** *If Conditions 1 and 3 hold then  $L_X^{\bullet} \leq_{st} L_Y^{\bullet}$ , or equivalently,  $\psi_X(u) \leq \psi_Y(u)$  for all  $u \geq 0$ .*

*Proof.* Since  $L_c^X \leq_{st} L_c^Y$  and  $N_X \leq_{st} N_Y$ , we have  $L_X^{\bullet} = \sum_{n=1}^{N_X} L_{c,n}^X \leq_{st} \sum_{n=1}^{N_Y} L_{c,n}^Y = L_Y^{\bullet}$  (Shaked & Shanthikumar (2007)). By Lemma 9 in the Appendix,  $\psi_X(u) = \Pr(L_X^{\bullet} > u) \leq \Pr(L_Y^{\bullet} > u) = \psi_Y(u)$  is reached.  $\square$

**LEMMA 5.** *If Condition 4 holds then  $\psi_{t,X} \leq_{s(n)} \psi_{t,Y} \Rightarrow \psi_{d,X} \leq_{s(n+1)} \psi_{d,Y}$ . Further, if  $D_X \geq D_Y$  holds then  $\psi_{d,1,X} \leq_{s(n+1)} \psi_{d,1,Y}$ .*

*Proof.* Since  $1 - \psi_t(u) = K * H(u) = (c/D) \int_0^u e^{-(c/D)(u-x)} K(x) dx$ , differentiation with respect to  $u$  gives  $-d\psi_t(u)/du = c\theta/[D(1 + \theta)]\psi_d(u)$  (Tsai (2003)). Then

$$\begin{aligned} \Pi_{\psi_t}^{(n)}(u) &= - \int_u^{\infty} (s-u)^n d\psi_t(s) = \frac{c\theta}{D(1 + \theta)} \int_u^{\infty} (s-u)^n \psi_d(s) ds \\ &= \frac{c\theta}{(n+1)D(1 + \theta)} \int_u^{\infty} \psi_d(s) d(s-u)^{n+1}. \end{aligned}$$

Integration by parts leads to

$$\Pi_{\psi_t}^{(n)}(u) = - \frac{c\theta}{(n+1)D(1 + \theta)} \int_u^{\infty} (s-u)^{n+1} d\psi_d(s) = \frac{c\theta}{(n+1)D(1 + \theta)} \Pi_{\psi_d}^{(n+1)}(u),$$

or equivalently,  $\Pi_{\psi_d}^{(n+1)}(u) = (n+1)\Pi_{\psi_t}^{(n)}(u)/\{(c/D)[\theta/(1 + \theta)]\}$ ,  $u \geq 0$ . Setting  $u=0$  gives

$$\int_0^{\infty} s^{n+1} d\psi_d(s) = -\Pi_{\psi_d}^{(n+1)}(0) = -(n+1)\Pi_{\psi_t}^{(n)}(0)/\frac{c\theta}{D(1 + \theta)} = (n+1) \int_0^{\infty} s^n d\psi_t(s)/\frac{c\theta}{D(1 + \theta)},$$

$n=0,1, \dots$ . Therefore, we conclude that  $[c_X/D_X][\theta_X/(1 + \theta_X)] \geq [c_Y/D_Y][\theta_Y/(1 + \theta_Y)]$  and  $\psi_{t,X} \leq_{s(n)} \psi_{t,Y}$  imply  $\psi_{d,X} \leq_{s(n+1)} \psi_{d,Y}$ . Assumption  $D_X \geq D_Y$  and condition 4 implies  $(1 + \theta_X)/(c_X\theta_X) \leq (1 + \theta_Y)/(c_Y\theta_Y)$ . By Eq. (2.2) and the argument in the proof to Theorem 2, we have  $\psi_{d,1,X} \leq_{s(n+1)} \psi_{d,1,Y}$ .  $\square$

**THEOREM 4.** *If Conditions 1, 2 and 3 hold then  $L_X^* \leq_{st} L_Y^*$ ,  $L_X \leq_{st} L_Y$  (or  $\bar{K}_X(u) \leq \bar{K}_Y(u)$ ,  $\psi_{t,X}(u) \leq \psi_{t,Y}(u)$  for all  $u \geq 0$ ) and  $\psi_{d,X} \leq_{st} \psi_{d,Y}$ . Further, if  $D_X \geq D_Y$  holds then  $\psi_{d,1,X} \leq_{st} \psi_{d,1,Y}$ .*

*Proof.* Since  $L_c^X \leq_{st} L_c^Y$ ,  $L_o^X \leq_{st} L_o^Y$  and  $N_X \leq_{st} N_Y$ , we have  $L_X^* = \sum_{n=1}^{N_X} (L_{o,n-1}^X + L_{c,n}^X) \leq_{st} \sum_{n=1}^{N_Y} (L_{o,n-1}^Y + L_{c,n}^Y) = L_Y^*$  (Shaked & Shanthikumar (2007)) and  $L_X = L_X^* + L_o^X \leq_{st} L_Y^* + L_o^Y = L_Y$ . By Lemma 9,  $\bar{K}_X(u) = \Pr(L_X^* > u) \leq \Pr(L_Y^* > u) = \bar{K}_Y(u)$  and  $\psi_{t,X}(u) = \Pr(L_X > u) \leq \Pr(L_Y > u) = \psi_{t,Y}(u)$ . As Conditions 2 and 3 imply Condition 4, and  $\psi_{t,X}(u) \leq \psi_{t,Y}(u)$

for all  $u \geq 0$  is equivalent to  $\psi_{t,X} \leq_{st} \psi_{t,Y}$  or  $\psi_{t,X} \leq_{sl(0)} \psi_{t,Y}$ , we conclude that  $\psi_{d,X} \leq_{sl} \psi_{d,Y}$  by Lemma 5, and  $\psi_{d,1,X} \leq_{sl} \psi_{d,1,Y}$  with additional condition  $D_X \geq D_Y$ .  $\square$

**COROLLARY 2.** *If Conditions 1, 2 and 3 satisfied are changed to  $X =_{mrl} Y$ ,  $c_X/D_X = c_Y/D_Y$  and  $\theta_X = \theta_Y$ , respectively, then  $\bar{K}_X(u) = \bar{K}_Y(u)$ ,  $\psi_{t,X}(u) = \psi_{t,Y}(u)$ ,  $\psi_{d,X}(u) = \psi_{d,Y}(u)$  and  $\psi_{s,X}(u) = \psi_{s,Y}(u)$  for all  $u \geq 0$ .*

*Proof.* Since  $X \leq_{mrl} Y \leq_{mrl} X$ ,  $c_X/D_X \leq c_Y/D_Y \leq c_X/D_X$  and  $\theta_X \leq \theta_Y \leq \theta_X$ , we have  $\bar{K}_X(u) \leq \bar{K}_Y(u) \leq \bar{K}_X(u)$  and  $\psi_{t,X}(u) \leq \psi_{t,Y}(u) \leq \psi_{t,X}(u)$ , implying  $\bar{K}_X(u) = \bar{K}_Y(u)$  and  $\psi_{t,X}(u) = \psi_{t,Y}(u)$ ; also,  $\psi_{d,X}(u) = [(1 + \theta_X)/\theta_X][\psi_{t,X}(u) - \bar{K}_X(u)] = [(1 + \theta_Y)/\theta_Y][\psi_{t,Y}(u) - \bar{K}_Y(u)] = \psi_{d,Y}(u)$ , and  $\psi_{s,X}(u) = \psi_{t,X}(u) - \psi_{d,X}(u) = \psi_{t,Y}(u) - \psi_{d,Y}(u) = \psi_{s,Y}(u)$ .  $\square$

**REMARKS:**

1. From Theorem 4, for the same individual claim size distribution function, bigger  $c$  (or larger  $\lambda$ ), smaller  $D$  (Condition 2), or larger  $\theta$  (Condition 3) will produce smaller  $\bar{K}(u)$  and  $\psi_t(u)$ , which is consistent with our intuition.
2. Condition  $X =_{mrl} Y$  in Corollary 2 implies  $E(X) = E(Y)$ ; combining this with  $c_X/D_X = c_Y/D_Y$  and  $\theta_X = \theta_Y$  gives  $\lambda_X/D_X = \lambda_Y/D_Y$ . Therefore, for the same individual claim size random variable  $X$ , increasing or decreasing both  $\lambda_X$  and  $D_X$  by an equal percentage keeps  $\bar{K}_X(u)$  and all ruin probabilities  $\psi_{t,X}(u)$ ,  $\psi_{d,X}(u)$  and  $\psi_{s,X}(u)$  unchanged.
3. Theorem 4 and Corollary 1 can be used to obtain lower/upper bounds for  $\bar{K}(u)$  and  $\psi_t(u)$ . If  $Y$  is the underlying risk ( $c_Y, D_Y, \lambda_Y, \theta_Y$  are given), and we want to find risks  $X$  and  $Z$  satisfying all conditions of Theorem 4 or Corollary 1; then  $\bar{K}_X(u) \leq \bar{K}_Y(u) \leq \bar{K}_Z(u)$  and  $\psi_{t,X}(u) \leq \psi_{t,Y}(u) \leq \psi_{t,Z}(u)$ . When  $X \leq_{sl} Y \leq_{sl} Z$  and  $E(X) = E(Y) = E(Z)$  are satisfied for Corollary 1, we set  $c_V = c_Y$ ,  $D_V = D_Y$ , and  $\lambda_V = \lambda_Y$ , (implying  $\theta_V = \theta_Y$ ) where  $V = X, Z$ ; when  $X \leq_{mrl} Y \leq_{mrl} Z$  are satisfied (implying  $E(X) \leq E(Y) \leq E(Z)$ ) for Theorem 4, setting  $c_V/D_V = c_Y/D_Y$  and  $\theta_V = \theta_Y$  (implying  $\lambda_V E(V)/D_V = \lambda_Y E(Y)/D_Y$ ) where  $V = X, Z$  can get tighter (larger) lower and tighter (smaller) upper bounds by Remark 1.
4. Theorem 4 and Corollary 1 have the same conclusions subject to different conditions. Conditions 2 and 3 for Theorem 4 are less restricted than conditions  $c_X = c_Y$ ,  $\lambda_X = \lambda_Y$ ,  $E(X) = E(Y)$  (implying  $\theta_X = \theta_Y$ ) and  $D_X = D_Y$  for Corollary 1; however, the relaxation must be compensated with a more restricted condition  $X \leq_{mrl} Y$  (for Theorem 4) changed from  $X \leq_{sl} Y$  (for Corollary 1) since  $X \leq_{mrl} Y \Rightarrow X \leq_{sl} Y$ . The following diagram gives the relationships for these conditions.

$$\begin{array}{ccc}
 \begin{array}{l} \text{condition for corollary 1} \\ \left\{ \begin{array}{l} c_X = c_Y, D_X = D_Y, E[X] = E[Y] \\ \theta_X = \theta_Y \end{array} \right. \\ X \leq_{sl} Y \end{array} & \begin{array}{c} \Rightarrow \\ \\ \Leftarrow \end{array} & \begin{array}{l} \text{conditions for Theorem 4} \\ \left\{ \begin{array}{l} c_X/D_X \geq c_Y/D_Y \\ \theta_X \geq \theta_Y \end{array} \right. \\ X \leq_{mrl} Y \end{array}
 \end{array}$$

5. Let  $\theta_X = \theta_Y$ ,  $c_X = c_Y$  (implying  $\lambda_X E(X) = \lambda_Y E(Y)$ ) and  $D_X = D_Y$  such that Conditions 2 and 3 are satisfied. If Condition 1 also holds, which implies  $E(X) = e_X(0) \leq$

$e_Y(0) = E(Y)$ , then high frequency and low severity risk  $X$  yields smaller  $\bar{K}(u)$  and  $\psi_t(u)$  than low frequency and high severity risk  $Y$  by Theorem 4. This gives theoretical proof to the question raised early in this section.

6. In general, it is difficult to get an expression for the survival function or its integration  $\int_t^\infty \bar{F}(u)du$  (e.g., Gamma, Lognormal and Weibull distributions), and hence the expression for the mean residual life function  $e(t)$  is not available. In this case, we cannot compare  $e_X(t)$  and  $e_Y(t)$  directly for satisfying Condition 1. However, since  $X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y$ , instead, we can compare risks  $X$  and  $Y$  with respect to likelihood ratio or hazard rate order. See Examples 5–8 in the appendix, in which cases  $X \leq_{lr} Y$  and hence Condition 1 is satisfied.

### 3. Ruin probability bounds from Erlang and Exponential claim amounts

In this section, we will give theorems for the orders of pair  $\bar{K}_X$  and  $\bar{K}_Y$ , pair  $\psi_{t,X}$  and  $\psi_{t,Y}$ , pair  $\psi_{d,X}$  and  $\psi_{d,Y}$ , and pair  $\psi_{d;1,X}$  and  $\psi_{d;1,Y}$  resulting from two claim size random variables  $X$  and  $Y$  distributed as a single Exponential, a mixture of Exponentials, Gamma, Pareto or Lognormal. The ordering relationships can be used to obtain upper and/or lower bounds on  $\bar{K}$  and  $\psi_t$ . Four examples are also given to illustrate the results of these theorems.

First, the following proposition shows that an exponentially distributed random variable is less, in the meaning of the first stop-loss order, than a random variable whose distribution function is a mixture of  $n$  exponentials with the same mean.

**PROPOSITION 1.** *Let  $\bar{F}_X(t) = e^{-\beta t}$  and  $\bar{F}_Y(t) = \sum_{i=1}^n q_i e^{-\beta_i t}$  where  $0 \leq q_i \leq 1$ ,  $i = 1, \dots, n$ , and  $\sum_{i=1}^n q_i = 1$ . Suppose  $E(X) = E(Y)$ ; then  $X \leq_{sl} Y$ .*

*Proof.* We need to show  $e^{-\beta t}/\beta = \Pi_X(t) \leq \Pi_Y(t) = \sum_{i=1}^n (q_i/\beta_i) e^{-\beta_i t}$  for all  $t \geq 0$  under the condition  $1/\beta = E(X) = E(Y) = \sum_{i=1}^n (q_i/\beta_i)$ . Using the fact that  $e^x - 1 \geq x$  for all real numbers  $x$ , we have  $\Pi_Y(t) - \Pi_X(t) = e^{-\beta t} \sum_{i=1}^n (q_i/\beta_i) (e^{(\beta - \beta_i)t} - 1) \geq e^{-\beta t} \sum_{i=1}^n (q_i/\beta_i) (\beta - \beta_i) = 0$ . □

Note that the proposition above extends Proposition 6 of Tsai (2006) in which case  $n = 2$ . Since low degree stop-loss order implies high degree stop-loss order (Goovaerts *et al.* (1990)), that is,  $X \leq_{sl(n)} Y \Rightarrow X \leq_{sl(m)} Y$  for  $n \leq m$ , we have in this case  $X \leq_{sl(n)} Y$ , which implies  $n!/\beta^n = E(X^n) \leq E(Y^n) = n! \sum_{i=1}^n q_i/\beta_i^n$ , for all  $n \geq 1$ .

With Proposition 1 and Theorem 1, we can extend Theorem 4 of Tsai (2006) to the following.

**THEOREM 5.** *Let  $\bar{F}_V(t) = e^{-\beta t}$  and  $\bar{F}_Y(t) = \sum_{i=1}^n q_i e^{-\beta_i t}$  where  $\sum_{i=1}^n q_i = 1$  and  $0 \leq q_i \leq 1$ ,  $i = 1, \dots, n$ . Suppose  $c_V = c_Y$ ,  $D_V = D_Y$ ,  $\lambda_V = \lambda_Y$  and  $E(V) = E(Y)$ ; then  $\bar{K}_V(u) \leq \bar{K}_Y(u)$  and  $\psi_{t,V}(u) \leq \psi_{t,Y}(u)$  for all  $u \geq 0$ ,  $\psi_{d,V} \leq_{sl} \psi_{d,Y}$  and  $\psi_{d;1,V} \leq_{sl} \psi_{d;1,Y}$ .*

Since likelihood ratio order implies mean residual lifetime order (see Diagram 1 in the Appendix), combining Example 8 (in the Appendix) and Theorem 4 gives the following theorem.

**THEOREM 6.** Let  $\bar{F}_Y(t) = \sum_{i=1}^n q_i e^{-\beta_i t}$  where  $\sum_{i=1}^n q_i = 1$  and  $0 \leq q_i \leq 1, i = 1, \dots, n, \bar{F}_X(t) = e^{-\max(\beta_1, \dots, \beta_n)t}$ , and  $\bar{F}_Z(t) = e^{-\min(\beta_1, \dots, \beta_n)t}$ . Suppose  $c_X/D_X \geq c_Y/D_Y \geq c_Z/D_Z$  and  $\theta_X \geq \theta_Y \geq \theta_Z$ ; then  $\bar{K}_X(u) \leq \bar{K}_Y(u) \leq \bar{K}_Z(u), \psi_{t,X}(u) \leq \psi_{t,Y}(u) \leq \psi_{t,Z}(u)$  for all  $u \geq 0$ , and  $\psi_{d,X} \leq {}_{sl}\psi_{d,Y} \leq {}_{sl}\psi_{d,Z}$ . If further  $D_X \geq D_Y \geq D_Z$  holds then  $\psi_{d,1,X} \leq {}_{sl}\psi_{d,1,Y} \leq {}_{sl}\psi_{d,1,Z}$ .

As mentioned in Remark 3, a tighter (larger) lower bound and a tighter (smaller) upper bound can be obtained by setting  $c_X/D_X = c_Z/D_Z = c_Y/D_Y$  and  $\theta_X = \theta_Z = \theta_Y$  (implying  $\lambda_X E(X)/D_X = \lambda_Z E(Z)/D_Z = \lambda_Y E(Y)/D_Y$ ). We wonder which of  $V$  of Theorem 5 or  $X$  of Theorem 6 produces tighter lower bounds for  $\bar{K}_Y(u)$  and  $\psi_{t,Y}(u)$ . First,  $c_X/D_X = c_Y/D_Y = c_V/D_V$  and  $\theta_X = \theta_Y = \theta_V$ . From  $1/\beta = E(V) = E(Y) = \sum_{i=1}^n q_i/\beta_i$ , we have  $e_X(t) = 1/\max(\beta_1, \dots, \beta_n) = \min(1/\beta_1, \dots, 1/\beta_n) = \sum_{i=1}^n q_i \min(1/\beta_1, \dots, 1/\beta_n) \leq \sum_{i=1}^n q_i/\beta_i = 1/\beta = e_V(t)$  for all  $t \geq 0$ . Therefore, we conclude that random variable  $V$  gives tighter bounds for both  $\bar{K}_Y(u)$  and  $\psi_{t,Y}(u)$  than random variable  $X$ , that is,  $\bar{K}_X(u) \leq \bar{K}_V(u) \leq \bar{K}_Y(u)$  and  $\psi_{t,X}(u) \leq \psi_{t,V}(u) \leq \psi_{t,Y}(u)$  for all  $u \geq 0$ .

From the insurer's, the insurance regulators' and policyholders' viewpoints, upper bounds on ruin probabilities are more important than lower bounds. Recall that ruin probability  $\psi_t(u)$  is roughly bounded above by  $e^{-Ru}$  where  $-R$  is the negative root of  $Ds^2 + cs + \lambda(\int_0^\infty e^{-su} dF(u) - 1) = 0$  and  $R$  is called the adjustment coefficient. Though asymptotical formulas can be approximated to ruin probabilities, all these approximations are good for large  $u$  only. Theorem 5 cannot offer upper bounds on both  $\bar{K}(u)$  and  $\psi_t(u)$  for the individual claim size random variables whose distribution function is a mixture of  $n$  exponentials; however, by Theorem 6 both lower and upper bounds on  $\bar{K}(u)$  and  $\psi_t(u)$  can be obtained. Tsai (2003) showed that explicit analytical solutions to  $\bar{K}, \psi_t, \psi_s$  and  $\psi_d$  can be obtained if the claim size distribution is a combination of exponentials or a mixture of Erlangs (Gammas with the first parameter set to be a positive integer). Though obtaining explicit analytical solutions to  $\bar{K}, \psi_t, \psi_s$  and  $\psi_d$  for a mixture of  $n$  exponentials is feasible, it involves solving a polynomial of degree  $n+1$  (Tsai (2003)); therefore, it takes time for large  $n$  to implement numerical methods to obtain these roots. Whereas, the upper and lower bounds in Theorem 6 obtained from single exponential claim size distributions  $Z$  and  $V$  (or  $X$ ), respectively, have formula-based expressions (Tsai (2006)), and hence they can be quickly and easily evaluated. Moreover, since Gamma  $(n, \beta) \leq_{mrl}$  Gamma  $(\alpha, \beta) \leq_{mrl}$  Gamma  $(n+1, \beta)$  by Example 5 in the Appendix where  $n$  is a positive integer with  $n < \alpha < n+1$ , from Theorem 4 we can obtain both lower and upper bounds on  $\bar{K}(u)$  and  $\psi_t(u)$  for Gamma  $(\alpha, \beta)$  if Conditions 2 and 3 are also satisfied, and these bounds have explicit analytical expressions. In this case, however, we cannot apply Corollary 1 to distributions Gamma  $(n, n\beta/\alpha)$ , Gamma  $(\alpha, \beta)$  and Gamma  $(n+1, (n+1)\beta/\alpha)$  with equal mean  $\alpha/\beta$  because there are no stop-loss ordering relationships between any two of them. These show that Theorem 4 has more applications than Corollary 1.

**EXAMPLE 1.** Upper and lower bounds on ruin probability for a mixture of two Exponentials

Table 1. Assumptions and roots  $s_1, s_2,$  and  $s_3$ .

	$q_1$	$q_2$	$\beta_1$	$\beta_2$	$\mu$	$\lambda$	$\theta$	$c$	$D$	$c/D$	$s_1$	$s_2$	$s_3$
$V$	1	0	1	-	1	1	0.2	1.2	0.5	2.4	0.12202662	3.27797338	-
$X$	1	0	2	-	0.5	2	0.2	1.2	0.5	2.4	0.19002488	4.20997512	-
$Y$	0.4	0.6	2	3/4	1	1	0.2	1.2	0.5	2.4	0.10800197	1.62658203	3.41541600
$Z$	0	1	-	3/4	4/3	3/4	0.2	1.2	0.5	2.4	0.09830606	3.05169394	-

Let  $\bar{F}(u) = \sum_{i=1}^n q_i e^{-\beta_i u}$  where  $\sum_{i=1}^n q_i = 1$  and  $0 \leq q_i \leq 1, i = 1, \dots, n$ . Tsai (2003) showed that the expressions for all of  $\bar{K}(u), \psi_t(u), \psi_d(u)$  and  $\psi_s(u)$  have the form  $\sum_{i=1}^{n+1} C_i e^{-s_i u}$  for some constants  $C_1, \dots, C_{n+1}$ ; see Tsai (2003) for detailed formulas. The corresponding  $m$ -th stop-loss transform is  $\Pi^{(m)}(u) = m! \sum_{i=1}^{n+1} (C_i / s_i^m) e^{-s_i u}, u \geq 0$ . To illustrate the results of Theorems 5 and 6 for  $n=2$  with an example, we consider four random variables  $V, X, Z$  (single exponential) and  $Y$  (a mixture of two exponentials) with equal  $\theta, c$  and  $D$ . The corresponding assumptions are listed in Table 1. Obviously,  $V \leq_{st} Y$  and  $X \leq_{mrl} Y \leq_{mrl} Z$ . From Figures 2 and 3, we observe as expected that

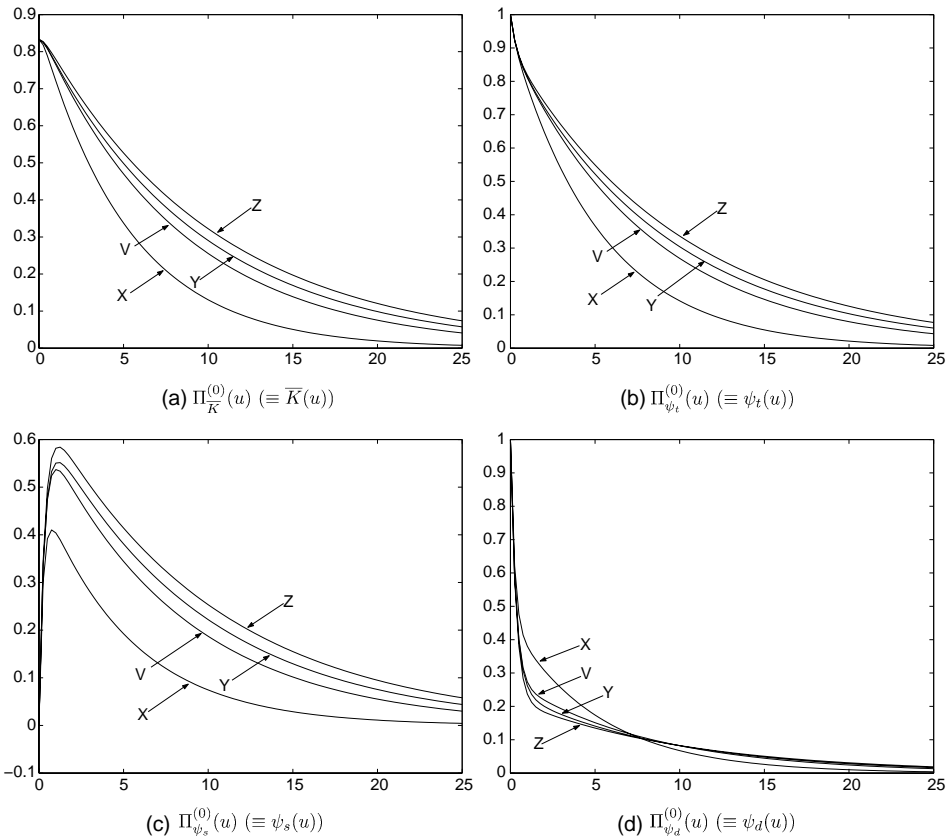


Figure 2. Ruin probabilities,  $\Pi^{(0)}(u)$ .

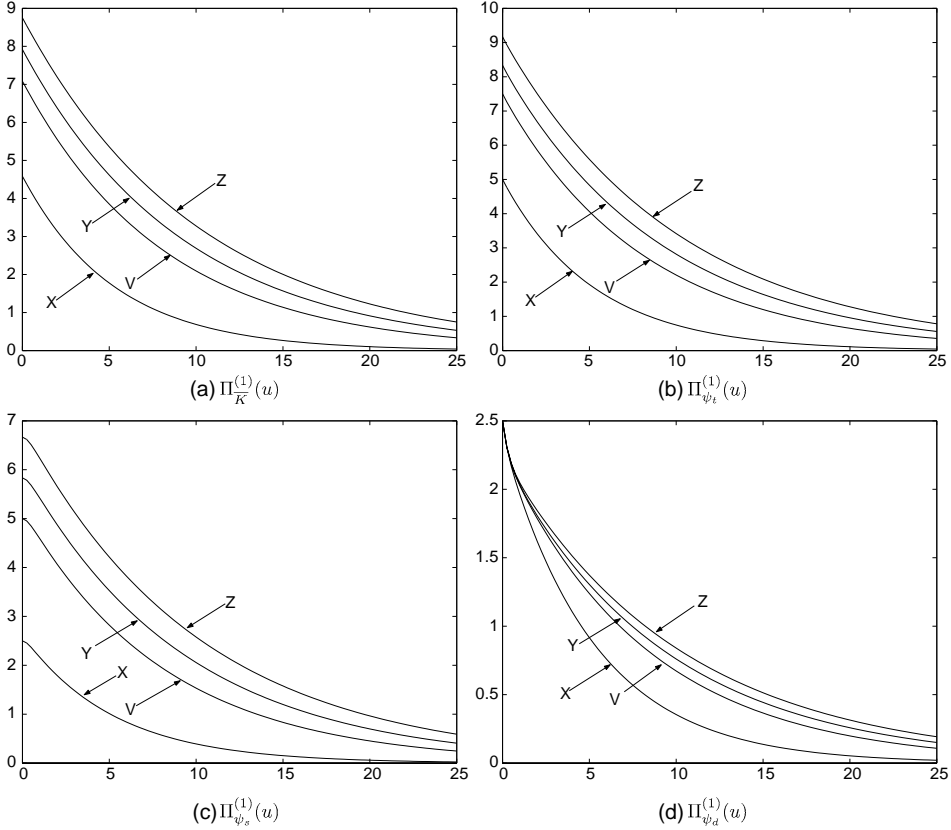


Figure 3. The first degree stop-loss transforms  $\Pi^{(1)}(u)$ .

- $\Pi_{\bar{K}_X}^{(m)}(u) \leq \Pi_{\bar{K}_Y}^{(m)}(u) \leq \Pi_{\bar{K}_V}^{(m)}(u) \leq \Pi_{\bar{K}_Z}^{(m)}(u)$  for  $m=0$  and all  $u \geq 0$ , implying it holds for  $m \geq 1$ ;
- $\Pi_{\psi_{t,X}}^{(m)}(u) \leq \Pi_{\psi_{t,V}}^{(m)}(u) \leq \Pi_{\psi_{t,Y}}^{(m)}(u) \leq \Pi_{\psi_{t,Z}}^{(m)}(u)$  for  $m=0$  and all  $u \geq 0$ , implying it holds for  $m \geq 1$ ; and
- $\Pi_{\psi_{d,X}}^{(m)}(u) \leq \Pi_{\psi_{d,V}}^{(m)}(u) \leq \Pi_{\psi_{d,Y}}^{(m)}(u) \leq \Pi_{\psi_{d,Z}}^{(m)}(u)$  does not hold for  $m=0$  and all  $u \geq 0$ , but it holds for  $m \geq 1$  and all  $u \geq 0$ .

Note that random variable  $V$  does give tighter bounds for  $\bar{K}_Y(u)$  and  $\psi_{t,Y}(u)$  than random variable  $X$ . Although  $\Pi_{\psi_{s,X}}^{(0)}(u) \leq \Pi_{\psi_{s,V}}^{(0)}(u) \leq \Pi_{\psi_{s,Y}}^{(0)}(u) \leq \Pi_{\psi_{s,Z}}^{(0)}(u)$  for all  $u \geq 0$  in this case, it is not true for some other cases even if Conditions 1, 2, and 3 are satisfied. For example, consider two single exponentials  $X$  and  $Y$  with assumptions given in Table 2. From Figure 4, we observe that  $\Pi_{\zeta_X}^{(m)}(u) \leq \Pi_{\zeta_Y}^{(m)}(u)$  for all  $u \geq 0$ ,  $m=0,1$ , and  $\zeta = \bar{K}, \psi_t, \psi_d$  (also for  $m=1$  and  $\zeta = \psi_s$ ), but  $\Pi_{\psi_{s,X}}^{(0)}(u) \geq \Pi_{\psi_{s,Y}}^{(0)}(u)$  for some  $u \geq 0$ .  $\square$

Table 2. Assumptions and roots  $s_1$  and  $s_2$ .

	$\beta$	$\mu$	$\lambda$	$\theta$	$c$	$D$	$c/D$	$s_1$	$s_2$
$X$	4	1/4	2	0.2	0.6	0.5	1.2	0.15868888	5.04131112
$Y$	2	1/2	1	0.2	0.6	2	0.3	0.04433278	2.25566722

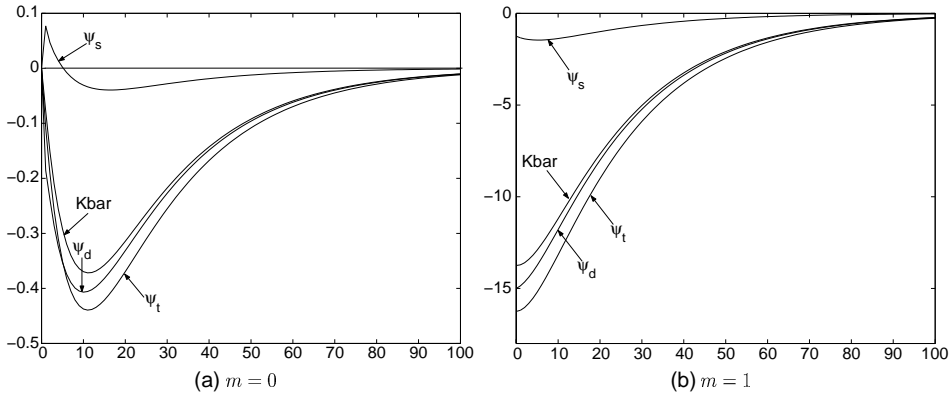


Figure 4. Difference between two stop-loss transforms  $\Pi_X^{(m)}(u) - \Pi_Y^{(m)}(u)$ .

**EXAMPLE 2.** Upper and lower bounds on ruin probability for Gamma distribution

Recall that both lower and upper bounds on  $\bar{K}(u)$  and  $\psi_i(u)$  with explicit analytical expressions can be obtained for Gamma  $(\alpha, \beta)$  claim size distribution with  $n < \alpha < n + 1$  where  $n$  is a positive integer if Conditions 2 and 3 are also satisfied since Gamma  $(n, \beta) \leq_{mrl}$  Gamma  $(\alpha, \beta) \leq_{mrl}$  Gamma  $(n + 1, \beta)$ . By Tsai (2003), the expressions for all of  $\bar{K}_Z(u)$ ,  $\psi_{t,Z}(u)$ ,  $\psi_{d,Z}(u)$  and  $\psi_{s,Z}(u)$  also have the form  $\sum_{i=1}^{n+1} C_i e^{-s_i u}$  for some constants  $C_1, \dots, C_{n+1}$ ; see Tsai (2003) for detailed expressions for  $s_i$  and  $C_i$ ,  $i = 1, 2, \dots, n + 1$ . To illustrate the upper and lower bounds on ruin probabilities for Gamma  $(\alpha, \beta)$  distributed claims, for simplicity, let  $X \sim \text{Gamma}(1, \beta) = \text{Exp}(\beta)$ ,  $Y \sim \text{Gamma}(\alpha, \beta)$  and  $Z \sim \text{Gamma}(2, \beta)$  ( $f_Z(t) = \beta^2 t e^{-\beta t}$ ) with equal  $\theta$ ,  $c$  and  $D$ . The corresponding assumptions are put in Table 3. Figure 5(a), (b) and (d) give upper and lower bounds on  $\bar{K}_Y(u)$ ,  $\psi_{t,Y}(u)$  and  $\Pi_{\psi_{d,Y}}^{(1)}(u)$ , respectively. Obviously and expectedly, Figure 5(c) does not offer upper and lower bounds on  $\Pi_{\psi_{d,Y}}^{(0)}(u) = \psi_{d,Y}(u)$  since there are crossing points between  $\psi_{d,X}(u)$  and  $\psi_{d,Z}(u)$ .  $\square$

Table 3. Assumptions and roots  $s_1$ ,  $s_2$  and  $s_3$ .

	$\alpha$	$\beta$	$\mu$	$\lambda$	$\theta$	$c$	$D$	$c/D$	$s_1$	$s_2$	$s_3$
$X$	1	3/4	4/3	3/4	0.2	1.2	0.5	2.4	0.09830606	3.05169394	-
$Y$	3/2	3/4	2	1/2	0.2	1.2	0.5	2.4	N/A	N/A	N/A
$Z$	2	3/4	8/3	3/8	0.2	1.2	0.5	2.4	0.07170161	1.18991661	2.63938178

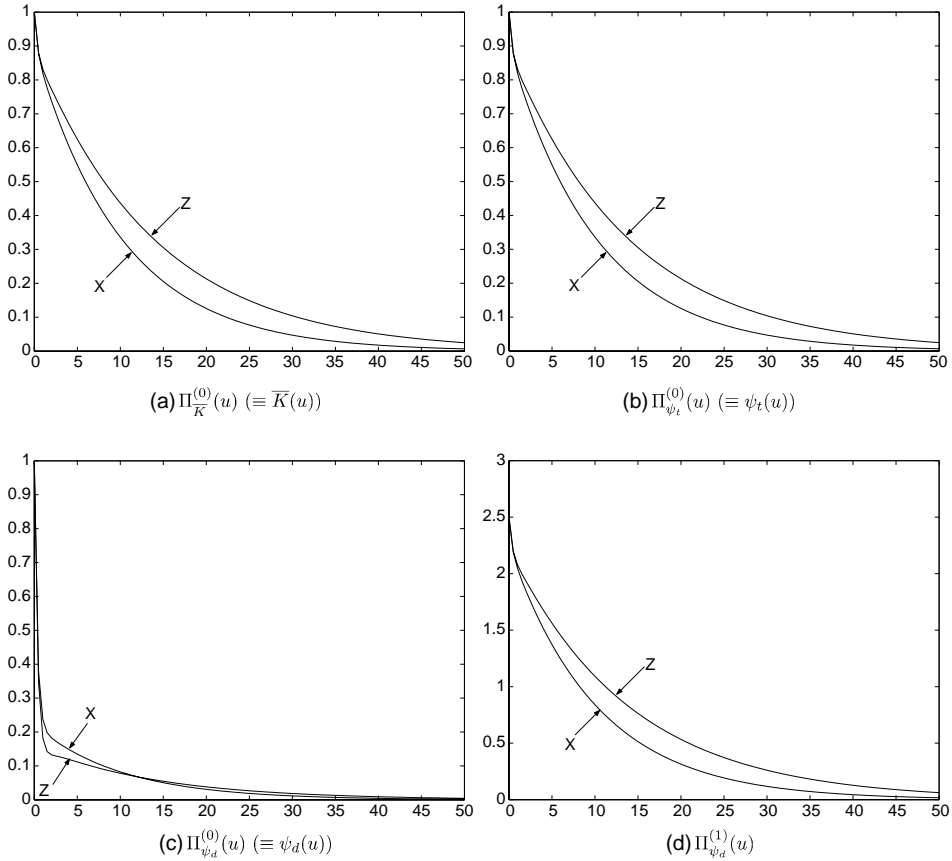


Figure 5. Upper and lower bounds for  $Y \sim \text{Gamma}(1.5, 0.75)$ .

The following two examples give lower bounds on ruin probabilities for Pareto and Lognormal distributed claims, respectively, and both have no explicit analytical expressions.

**EXAMPLE 3.** Lower bounds on ruin probability for Pareto distribution

Let  $W \sim \text{Pareto}(\tau, \theta)$  ( $\tau > 1$ ) with the survival function  $\bar{F}_W(t) = [\theta/(t + \theta)]^\tau$ . The mean residual lifetime function of  $W$  can be easily derived as  $e_W(t) = (t + \theta)/(\tau - 1)$  which is strictly increasing in  $t$  from its mean  $E[W] = \theta/(\tau - 1)$  to  $\infty$ . The mean residual lifetime function of  $Z$ , which is Gamma  $(\alpha, \beta)$  distributed, is strictly increasing (decreasing) from its mean  $\alpha/\beta$  to  $1/\beta$  for  $\alpha < (>) 1$  (Klugman *et al.* (2004), p. 53), and  $\text{Exp}(\beta)$  has a constant mean residual lifetime function  $1/\beta$ . Therefore, if  $\alpha \leq 1$  and  $(1/\beta) \leq [\theta/(\tau - 1)]$  then  $e_Z(t) \leq e_W(t)$  for all  $t \geq 0$ ; if  $\alpha \geq 1$  and  $E[\text{Gamma}(\alpha, \beta)] = (\alpha/\beta) \leq [\theta/(\tau - 1)] = E[\text{Pareto}(\tau, \theta)]$  then  $e_Z(t) \leq e_W(t)$  for all  $t \geq 0$ , implying  $\text{Gamma}(\alpha, \beta) \leq_{mrl} \text{Pareto}(\tau, \theta)$ . For

example,  $Z \sim \text{Gamma}(2, 3/4) \leq_{mrl} \text{Pareto}(\tau, \theta)$  if  $\theta/(\tau - 1) \geq 8/3$ . Feasible lower bounds from  $Z$  on  $\bar{K}_W(u)$ ,  $\psi_{t,W}(u)$  and  $\Pi_{\psi_{d,W}}^{(1)}(u)$  are given in Figure 5(a), (b) and (d), respectively.

**PROPOSITION 2.** *A feasible lower bound resulting from Gamma  $(n, \beta)$  on ruin probability for Pareto  $(\tau, \theta)$  distributed claims with  $\tau > 1$  can be obtained if  $1 \leq n \leq \beta[\theta/(\tau - 1)]$ .*

Similarly, let  $Y$  have survival function  $\bar{F}_Y(u) = \sum_{i=1}^n q_i e^{-\beta_i u}$  where  $\sum_{i=1}^n q_i = 1$  and  $0 \leq q_i \leq 1$ ,  $i = 1, \dots, n$ , and  $\bar{F}_Z(u) = e^{-\min(\beta_1, \dots, \beta_n)u}$ . If  $E[Z] = 1/\min(\beta_1, \dots, \beta_n) \leq [\theta/(\tau - 1)] = E[W]$  then  $e_Z(t) \leq e_W(t)$  for all  $t \geq 0$ , implying  $Z \leq_{mrl} W$  and thus  $Y \leq_{mrl} Z \leq_{mrl} W$  by Example 8 in the Appendix. Given assumptions in Table 1 for  $Y$  and  $Z$ , if  $\theta/(\tau - 1) \geq 4/3$  then feasible lower bounds from  $Z$  (tighter than the ones from  $Y$ ) on  $\bar{K}_W(u)$ ,  $\psi_{t,W}(u)$  and  $\Pi_{\psi_{d,W}}^{(1)}(u)$  are given in Figure 2(a), (b) and (d), respectively.  $\square$

**EXAMPLE 4.** Lower bounds on ruin probability for Lognormal distribution

Let  $W \sim \text{Lognormal}(\mu, \sigma)$  ( $\mu$  can be negative) with the probability density function  $f_W(t) = ((e^{-1/2[(\ln t - \mu)/\sigma]^2})/(t\sigma\sqrt{2\pi}))$ , and  $Z \sim \text{Gamma}(\alpha, \beta)$  with  $f_Z(t) = ((\beta t)^\alpha e^{-\beta t})/(t\Gamma(\alpha))$ . Since there are no explicit analytical expressions for the survival functions of  $Z$  and  $W$ , we would compare  $Z$  and  $W$  with respect to the likelihood ratio order rather than the hazard rate order or the mean residual lifetime order. By Definition 1 in the Appendix,  $Z \leq_{lr} (\geq_{lr}) W$  if  $h(t) = f_Z(t)/f_W(t) = Ct^\alpha e^{[(\ln t - \mu)/\sigma]^2/2 - \beta t}$  is non-increasing (non-decreasing) in  $t$  where  $C = \beta^\alpha \sigma \sqrt{2\pi}/\Gamma(\alpha)$ , or  $h'(t) = -Ct^{\alpha-1} e^{1/2[(\ln t - \mu)/\sigma]^2 - \beta t} [(\beta \sigma t + \mu - \alpha \sigma) - \ln t]/\sigma \leq (\geq) 0$  for all  $t > 0$ . Since any linear function with positive slope goes to infinity faster than the natural logarithm, it is impossible that  $(\beta \sigma t + \mu - \alpha \sigma) - \ln t \leq 0$  for all  $t > 0$ . Thus, we cannot find any pair of parameters  $(\alpha, \beta)$  such that  $\text{Lognormal}(\mu, \sigma) \leq_{lr} \text{Gamma}(\alpha, \beta)$ . Because  $e^{-(\mu - \alpha \sigma + 1)t} + (\mu - \alpha \sigma)$  is the tangent to  $\ln t$  at  $t = e^{\mu - \alpha \sigma + 1}$ , we have that if  $\beta \sigma \geq e^{-(\mu - \alpha \sigma + 1)}$  (or  $\alpha \leq [\mu + 1 + \ln(\beta \sigma)]/\sigma$ ) then  $\beta \sigma t + (\mu - \alpha \sigma) \geq e^{-(\mu - \alpha \sigma + 1)t} + (\mu - \alpha \sigma) \geq \ln t$  for all  $t > 0$ , which implies  $\text{Gamma}(\alpha, \beta) \leq_{lr} \text{Lognormal}(\mu, \sigma)$ . Thus,  $\text{Gamma}(\alpha, \beta) \leq_{mrl} \text{Lognormal}(\mu, \sigma)$  by Diagram 1 in the Appendix. For example,  $Z \sim \text{Gamma}(2, 3/4) \leq_{mrl} \text{Lognormal}(\mu, \sigma)$  if  $(3/4)\sigma \geq e^{-(\mu - 2\sigma + 1)}$ . Feasible lower bounds from  $Z$  on  $\bar{K}_W(u)$ ,  $\psi_{t,W}(u)$  and  $\Pi_{\psi_{d,W}}^{(1)}(u)$  are given in Figure 5(a), (b) and (d), respectively.

**PROPOSITION 3.** *A feasible lower bound resulting from Gamma  $(n, \beta)$  on ruin probability for Lognormal  $(\mu, \sigma)$  distributed claims can be obtained if  $1 \leq n \leq [\mu + 1 + \ln(\beta \sigma)]/\sigma$ .*  $\square$

Note that to obtain tighter lower bounds resulting from Gamma  $(n, \beta)$  on ruin probabilities for both Pareto  $(\tau, \theta)$  and Lognormal  $(\mu, \sigma)$  distributed claims, we should take larger integers  $n$  satisfying  $1 \leq n \leq \beta[\theta/(\tau - 1)]$  and  $1 \leq n \leq [\mu + 1 + \ln(\beta \sigma)]/\sigma$ , respectively. However, bigger  $n$  leads to more time and efforts to implement numerical methods to obtain  $n+1$  roots,  $s_1, s_2, \dots, s_{n+1}$ , to a polynomial of degree  $n+1$  for ruin probability  $\psi_t(u) = \sum_{i=1}^{n+1} C_i e^{-s_i u}$ .

## Acknowledgements

The author would like to thank the referee for their valuable comments and suggestions. Support from the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged.

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## Appendix

### A technical preliminary

In the appendix, we propose some definitions and lemmas for a variety of orders between two random variables (Shaked & Shanthikumar (2007)). In addition, we give some examples for the likelihood ratio order between two random variables used in this paper.

First, let  $r_Z(t) = f_Z(t)/\bar{F}_Z(t)$  be the hazard rate (or failure rate) function,  $e_Z(t) = E(Z - t | Z > t) = \int_t^\infty \bar{F}_Z(u) du / \bar{F}_Z(t)$  be the mean residual lifetime function, and  $\Pi_Z^{(k)}(t) = \int_t^\infty (u - t)^k dF_Z(u)$  be the  $k$ -th stop-loss transform,  $k = 0, 1, \dots$ , for a random variable  $Z$ . When  $k = 1$ , the first stop-loss transform is denoted by  $\Pi_Z(t)$ .

**DEFINITION 1.**  $X$  is less than  $Y$  in the meaning of the likelihood ratio order, denoted by  $X \leq_{lr} Y$ , if  $dF_X(t)/dF_Y(t)$  is non-increasing in  $t$  over  $\{t: dF_Y(t) > 0\}$ .

**LEMMA 6.**  $X \leq_{lr} Y \Leftrightarrow dF_X(s)dF_Y(t) \geq dF_X(t)dF_Y(s)$  for all  $s \leq t$ .

**DEFINITION 2.**  $X$  is less than  $Y$  in the meaning of the hazard rate order, denoted by  $X \leq_{hr} Y$ , if  $r_X(t) \geq r_Y(t)$  for all  $t \geq 0$ .

**LEMMA 7.**  $X \leq_{hr} Y \Leftrightarrow \bar{F}_X(t)/\bar{F}_Y(t)$  is non-increasing in  $t$  over  $\{t: \bar{F}_Y(t) > 0\} \Leftrightarrow \bar{F}_X(s)\bar{F}_Y(t) \geq \bar{F}_X(t)\bar{F}_Y(s)$  for all  $s \leq t$ .

**DEFINITION 3.**  $X$  is less than  $Y$  in the meaning of the mean residual lifetime order, denoted by  $X \leq_{mrl} Y$ , if  $e_X(t) \leq e_Y(t)$  for all  $t \geq 0$ .

**LEMMA 8.**  $X \leq_{mrl} Y \Leftrightarrow \int_t^\infty \bar{F}_X(u)du / \int_t^\infty \bar{F}_Y(u)du$  is non-increasing in  $t$  over  $\{t: \int_t^\infty \bar{F}_Y(u)du > 0\} \Leftrightarrow [\int_s^\infty \bar{F}_X(u)du][\int_t^\infty \bar{F}_Y(u)du] \geq [\int_t^\infty \bar{F}_X(u)du][\int_s^\infty \bar{F}_Y(u)du]$  for all  $s \leq t$ .

**DEFINITION 4.**  $X$  is less than  $Y$  in the meaning of the stochastic order, denoted by  $X \leq_{st} Y$ , if  $E[w(X)] \leq E[w(Y)]$  for all non-decreasing real function  $w$  on  $[0, \infty)$ .

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow \begin{matrix} X \leq_{mrl} Y \\ X \leq_{st} Y \end{matrix} \Rightarrow X \leq_{sl} Y$$

**LEMMA 9.**  $X \leq_{st} Y \Leftrightarrow \bar{F}_X(t) \leq \bar{F}_Y(t)$  for all  $t \geq 0$ .

The following implications of the orderings of a pair of random variables  $X$  and  $Y$  can be found from Shaked & Shanthikumar (2007).

**DIAGRAM 1.**

There is no implication relationship between stochastic order and the mean residual lifetime order. However, if  $e_X(t)/e_Y(t)$  is non-decreasing in  $t$ , then  $X \leq_{mrl} Y \Rightarrow X \leq_{hr} Y$  which implies  $X \leq_{st} Y$  (Shaked & Shanthikumar (2007)).

**DEFINITION 5.**  $X$  is less than  $Y$  in the meaning of the  $n$ -th stop-loss order, denoted by  $X \leq_{sl(n)} Y$ ,  $n = 0, 1, 2, \dots$ , if  $E(X^k) \leq E(Y^k)$ ,  $k = 1, 2, \dots, n - 1$ , and  $\Pi_X^{(n)}(t) \leq \Pi_Y^{(n)}(t)$  for all  $t \geq 0$ .

Note that  $X \leq_{sl(0)} Y$  is equivalent to  $X \leq_{st} Y$  since  $\Pi^{(0)}(t) = \bar{F}(t)$ , and we denote  $X \leq_{sl} Y$  for  $n = 1$ . The  $k$ -th stop-loss transform can be generalized for a non-negative and decreasing function which is not necessarily a distribution tail as follows: let  $\Omega = \{Q(u), u \geq 0: Q(u) \geq 0, \text{ decreasing and } \lim_{u \rightarrow \infty} Q(u) = 0\}$ ; the  $k$ -th stop-loss transform  $\Pi_Q^{(k)}(t)$  of  $Q \in \Omega$  is defined as  $\Pi_Q^{(k)}(t) = -\int_t^\infty (u - t)^k dQ(u)$ . With the definition of  $\Omega$ , Cheng & Pai (2003) generalized Definition 5 to the following.

**DEFINITION 6.** Suppose  $Q_1, Q_2 \in \Omega$  and the  $n$ -th stop-loss transforms of  $Q_1$  and  $Q_2$  exist.  $Q_1$  is less than  $Q_2$  in the meaning of the  $n$ -th stop-loss order, denoted by  $Q_1 \leq_{sl(n)} Q_2$ , if

$$-\int_0^\infty u^k dQ_1(u) \leq -\int_0^\infty u^k dQ_2(u), \quad k = 1, 2, \dots, n-1, \quad \text{and} \quad -\int_t^\infty (u-t)^n dQ_1(u) = \Pi_{Q_1}^{(n)}(t) \leq \Pi_{Q_2}^{(n)}(t) = -\int_t^\infty (u-t)^n dQ_2(u) \text{ for all } t \geq 0.$$

**EXAMPLE 5.** Let  $X \sim \text{Gamma}(\alpha_1, \beta_1)$ ,  $Y \sim \text{Gamma}(\alpha_2, \beta_1)$  and  $Z \sim \text{Gamma}(\alpha_2, \beta_2)$ . Then  $X \leq_{lr} Y \leq_{lr} Z \Leftrightarrow 0 < \alpha_1 \leq \alpha_2$  and  $0 < \beta_2 \leq \beta_1$ .

*Proof.*  $X \leq_{lr} Y \Leftrightarrow dF_X(t)/dF_Y(t) = \{[(\beta_1 t)^{\alpha_1} e^{-\beta_1 t}]/[\Gamma(\alpha_1)]\} \{[\Gamma(\alpha_2)]/[(\beta_1 t)^{\alpha_2} e^{-\beta_1 t}]\} = [\Gamma(\alpha_2)/\Gamma(\alpha_1)](\beta_1 t)^{\alpha_1 - \alpha_2}$  is non-increasing in  $t \Leftrightarrow 0 < \alpha_1 \leq \alpha_2$ ;  $Y \leq_{lr} Z \Leftrightarrow dF_Y(t)/dF_Z(t) = \{[(\beta_1 t)^{\alpha_2} e^{-\beta_1 t}]/[\Gamma(\alpha_2)]\} \{[\Gamma(\alpha_2)]/[(\beta_2 t)^{\alpha_2} e^{-\beta_2 t}]\} = (\beta_1/\beta_2)^{\alpha_2} e^{-(\beta_1 - \beta_2)t}$  is non-increasing in  $t \Leftrightarrow 0 < \beta_2 \leq \beta_1$ .  $\square$

**EXAMPLE 6.** Let  $X \sim \text{Geometric}(\beta_1)$  and  $Y \sim \text{Geometric}(\beta_2)$ . Then  $X \leq_{lr} Y \Leftrightarrow 0 < \beta_1 \leq \beta_2$ .

*Proof.*  $X \leq_{lr} Y \Leftrightarrow dF_X(k)/dF_Y(k) = \{[1/(1 + \beta_1)][\beta_1/(1 + \beta_1)]^k\} / \{[1/(1 + \beta_2)][\beta_2/(1 + \beta_2)]^k\} [(1 + \beta_2)/(1 + \beta_1)] \{[\beta_1(1 + \beta_2)]/[\beta_2(1 + \beta_1)]\}^k$  is non-increasing in  $k \Leftrightarrow [\beta_1(1 + \beta_2)]/[\beta_2(1 + \beta_1)] \leq 1 \Leftrightarrow 0 < \beta_1 \leq \beta_2$ .  $\square$

**EXAMPLE 7.** Let  $\bar{F}_X(t) = e^{-\beta t}$  and  $\bar{F}_Y(t) = \sum_{i=1}^n q_i e^{-\beta_i t}$  where  $0 \leq q_i \leq 1$ ,  $i = 1, \dots, n$ , and  $\sum_{i=1}^n q_i = 1$ . Then  $\beta \geq \max(\beta_1, \dots, \beta_n) \Rightarrow X \leq_{lr} Y$ , and  $\beta \leq \min(\beta_1, \dots, \beta_n) \Rightarrow X \geq_{lr} Y$ .

*Proof.*  $(dF_Y(t)/dF_X(t)) = \sum_{i=1}^n q_i (\beta_i/\beta) e^{-(\beta_i - \beta)t}$  is non-increasing in  $t$  if  $\beta_i \geq \beta$ ,  $i = 1, \dots, n$ , and non-decreasing in  $t$  if  $\beta_i \leq \beta$ ,  $i = 1, \dots, n$ . Therefore,  $X \leq_{lr} Y$  for  $\beta \geq \max(\beta_1, \dots, \beta_n)$ , and  $X \geq_{lr} Y$  for  $\beta \leq \min(\beta_1, \dots, \beta_n)$ .  $\square$

**EXAMPLE 8.** Let  $\bar{F}_Y(t) = \sum_{i=1}^n q_i e^{-\beta_i t}$  where  $0 \leq q_i \leq 1$ ,  $i = 1, \dots, n$ , and  $\sum_{i=1}^n q_i = 1$ ,  $\bar{F}_X(t) = e^{-\max(\beta_1, \dots, \beta_n)t}$ , and  $\bar{F}_Z(t) = e^{-\min(\beta_1, \dots, \beta_n)t}$ . Then  $X \leq_{lr} Y \leq_{lr} Z$ .