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Original Article

An effective method for constructing bounds for ruin probabilities for the surplus process perturbed by diffusion

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In this paper, we first study orders, valid up to a certain positive initial surplus, between a pair of ruin probabilities resulting from two individual claim size random variables for corresponding continuous time surplus processes perturbed by diffusion. The results are then applied to obtain a smooth upper (lower) bound for the underlying ruin probability; the upper (lower) bound is constructed from exponentially distributed claims, provided that the mean residual lifetime function of the underlying random variable is non-decreasing (non-increasing). Finally, numerical examples are given to illustrate the constructed upper bounds for ruin probabilities with comparisons to some existing ones.

Keywords: Surplus process; Diffusion process; Ruin probability; Maximal aggregate loss; Compound geometric distribution; Ordering; Bound

1. Introduction

The nature of the insurance business is the promise by the insurer to pay all claims of the insured that are covered in exchange for a policy premium. In addition, the insurer commits some capital (surplus) to assure that the promise will be kept even under special circumstances. Ruin probability is one of solvency-dependent measures for financial soundness of the insurer. To study the ruin probability, consider the classical continuous time surplus process perturbed by diffusion at time t,

$$U_Z(t) = u + c_Z t - S_Z(t) + \sigma_Z W(t), \quad t \geq 0, \tag{1.1}$$

where c_Z is the constant premium rate received per unit time, $u = U_Z(0)$ is the initial surplus, $\sigma_Z > 0$ (σ_Z^2 is called the variance parameter), and $\{W(t) : t \geq 0\}$ is a standard Wiener process. Moreover, $S_Z(t) = Z_1 + Z_2 + \cdots + Z_{N_Z(t)}$ is the aggregate claims up to the time t; the number of claims up to time t, $N_Z(t)$, is assumed to follow a Poisson process with parameter λ_Z ($S_Z(t) = 0$ if $N_Z(t) = 0$); the individual claim sizes $Z_1, Z_2, \ldots$, independent of $N_Z(t)$ and $W(t)$, are positive, independent, and identically distributed

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random variables (as Z) with common distribution function $F_Z(t)=\Pr(Z\le t)$. We assume $c_Z=\lambda_Z E[Z](1+\theta_Z)$ where $\theta_Z>0$ is the relative security loading.

Let $T=\inf\{t{:}U_Z(t)\le 0\}$ ($T=\infty$ if the set is empty) be the time of ruin. Define

$$\psi_{d,Z}(u)=\Pr(T<\infty,U_Z(T)=0|U_Z(0)=u)$$

to be the probability of ruin caused by oscillation, and

$$\psi_{s,Z}(u)=\Pr(T<\infty,U_Z(T)<0|U_Z(0)=u)$$

to be the probability of ruin caused by a claim; then

$$\psi_{t,Z}(u)=\psi_{d,Z}(u)+\psi_{s,Z}(u)=\Pr(T<\infty|U_Z(0)=u)$$

is the probability of ruin. It was showed in Dufresne & Gerber (1991) that $\psi_{t,z}(u)=\Pr(L_z>u)$, the tail probability of the maximal aggregate loss random variable is defined by $L_Z=\max\{u–U_Z(t):t\ge 0\}$. In fact L_Z can be decomposed as:

$$L_Z=L^Z_{o,0}+L^Z_{c,1}+L^Z_{o,1}+\ldots+L^Z_{c,N_Z}+L^Z_{o,N_Z}=\sum_{n=1}^{N_Z}(L^Z_{o,n-1}+L^Z_{c,n})+L^Z_{o,N_Z}$$

with $L_Z=L^Z_{o,0}$ if $N_Z=0$, where $L^Z_{o,n}$ and $L^Z_{c,n}$ are the amounts that result in the $(n+1)$th and nth record highs of the aggregate loss process $\{u–U_Z(t):t\ge 0\}$ due to oscillation and a claim, respectively, and N_Z, the number of record highs of the process $\{u–U_Z(t){:}t\ge 0\}$ caused by a claim, follows the geometric distribution with mean $1/\theta_Z$. The random variables $L^Z_{o,0},L^Z_{o,1},L^Z_{o,2},\ldots$ are identically distributed (as L^Z_o) with common distribution function $H_Z(u)=1-e^{-(c_Z/D_Z)u}$ where $D_Z=\sigma^2_Z/2$, and $L^Z_{c,1},L^Z_{c,2},L^Z_{c,3},\ldots$ are identically distributed (as L^Z_c) with common distribution function $\Gamma_Z(u)=\int_0^u\overline{F}_Z(t)dt/E[Z]$. In addition, N_Z, $L^Z_{o,0}$, $L^Z_{c,1}$, $L^Z_{o,1}$, $L^Z_{c,2}$, $L^Z_{o,2},\ldots$ are independent. Moreover, Tsai (2003) showed that

$$\overline{K}_Z(u)\overset{\Delta}{=}\Pr(L^\star_Z>u)=\frac{1}{1+\theta_Z}\psi_{d,Z}(u)+\psi_{s,Z}(u),\quad u\ge 0,$$

where

$$L^\star_Z=L^Z_{o,0}+L^Z_{c,1}+\ldots+L^Z_{o,N-1}+L^Z_{c,N}=\sum_{n=1}^{N_Z}(L^Z_{o,n-1}+L^Z_{c,n}).\tag{1.2}$$

Since $\overline{K}_Z(u)$ is the tail probability of $L^\star_Z$, $\overline{K}_Z(u)$ can be further expressed as a compound geometric tail distribution with mean $1/\theta_Z$, i.e.

$$\overline{K}_Z(u)=\sum_{n=1}^{\infty}\frac{\theta_Z}{1+\theta_Z}\left[\frac{1}{1+\theta_Z}\right]^n\overline{G^{*n}_Z}(u),\quad u\ge 0,\tag{1.3}$$

with $\overline{K}_Z(0)=1/(1+\theta_Z)$, where $G_Z(y)=H_Z*\Gamma_Z(y)=\int_0^y H_Z(y-t)d\Gamma_Z(t)$ is the distribution function of $L^Z_o+L^Z_c$. Since $L_Z=L^\star_Z+L^Z_o$, we have that $\psi_{t,Z}(u)=\Pr(L_Z>u)$, the ruin probability for surplus process (1.1), is actually the convolution of two distribution functions, given by

$$\psi_{t,Z}(u) = \overline{K_Z * H_Z}(u) = \sum_{n=0}^{\infty} \frac{\theta_Z}{1+\theta_Z}\left[\frac{1}{1+\theta_Z}\right]^n \overline{G_Z^{*n} * H_Z}(u), \quad u \geq 0. \tag{1.4}$$

When the diffusion component is removed from Eq. (1.1), that is, $\sigma_Z = 0$, surplus process in Eq. (1.1) reduces to:

$$U_Z(t) = u + c_Z t - S_Z(t), \quad t \geq 0. \tag{1.5}$$

In this case, $L_{o,n}^Z = 0$, $n = 0, 1, \ldots,$ and $L_Z = L_Z^\star = \Sigma_{n=1}^{N_Z} L_{c,n}^Z \triangleq L_Z^\bullet$. Accordingly, both $\psi_{t,Z}(u)$ and $\overline{K}_Z(u)$ reduce to $\psi_Z(u) = \Pr(T < \infty | U_Z(0) = u)$, the ruin probability for surplus process (1.5), and expressions (1.3) and (1.4) simplify to:

$$\psi_Z(u) = \Pr(L_Z^\bullet > u) = \sum_{n=1}^{\infty} \frac{\theta_Z}{1+\theta_Z}\left[\frac{1}{1+\theta_Z}\right]^n \overline{\Gamma_Z^{*n}}(u), \quad u \geq 0, \tag{1.6}$$

which is also a compound geometric tail distribution with mean $1/\theta_Z$ and $\psi_Z(0) = 1/(1+\theta_Z)$.

In general, explicit expressions of the ruin probability (1.4) and (1.6) are unavailable. Only in few cases, for instance, when the claims are exponentially distributed we succeed in getting explicit formulas. Therefore, the derivation of formula-based bounds or approximations for the ruin probabilities becomes very important for the insurance business. As pointed out by Rolski *et al.* (1999), a large number of Lundberg bounds have been derived; see their Bibliographical Notes of Section 5.4. Other examples of the formula-based bounds for $\psi_Z(u)$ can be found in Broeckx *et al.* (1986), Willmot (1994), and Cai & Garrido (1998), and the references therein. For formula-based approximation to $\psi_Z(u)$, see Beekman (1969), De Vylder (1978), and Grandell (2000).

The aim of this paper is to construct smooth upper or lower bounds for the ruin probabilities by applying the results from the ordering of the two ruin functions for surplus processes in Eqs. (1.1) and (1.5), respectively; these bounds are obtained from the corresponding surplus process with exponential claim amount distributions and valid only up to a certain positive initial surplus. As the ruin probability for surplus process (1.1) or (1.5), with exponentially distributed claim amounts has a closed-form expression, our smooth bounds for the ruin probabilities are formula-based and can be easily computed. The idea of using exponential distribution for the claim amount random variables was also presented by De Vylder (1978) where the so-called De Vylder's approximation to the ruin probability was introduced; the ruin probability of the underlying surplus process is approximated by one of the surplus process with exponentially distributed claims by matching the first three moments of these two surplus processes.

When there is an ordering relationship between two ruin functions, the bigger (smaller) one can be served as an upper (lower) bound of the smaller (larger) ruin function. Previously published articles focused on $\psi_{t,X}(u) \leq \psi_{t,Y}(u)$ or $\psi_X(u) \leq \psi_Y(u)$ for all $u \geq 0$ under some conditions; see, for example, Daley & Rolski (1984), Cheng & Pai (2003), and Tsai (2006, 2008, 2009). One of these conditions regards some orderings of claim size random variables X and Y, for instance, $X \leq_{mrl} Y$ or $X \leq_{sl} Y$ which holds in general only if X and Y come from the same distribution family with different parameters (see Example 3 in Tsai (2008)). The ordering results derived in this paper apply when X and Y are from different distribution families which the insurer encounters in most cases.

In Section 2 of this paper, we first show $\overline{K}_X(u) \leq \overline{K}_Y(u)$, $\psi_{t,X}(u) \leq \psi_{t,Y}(u)$, and $\psi_X(u) \leq \psi_Y(u)$, $u \in [0, u_*]$, for some $u_* > 0$ under some conditions including $X \leq_{hmrl\ [0,u_*]} Y$ (see Definition 4 in Appendix A). In Section 3, we propose an effective method to construct a smooth upper (lower) bound on each of $\overline{K}_X$ and ψ_X provided that the mean residual lifetime function $e_X(t)$ is non-decreasing (non-increasing) in $t \in [0, u_*]$. The constructed upper bound is then compared with the Lundberg bound and the ones proposed by Broeckx *et al.* (1986), Willmot (1994), and Cai & Garrido (1998) with numerical examples for illustration. In Appendix A, we give definitions for a variety of orders, valid over some interval between two random variables X and Y, for convenience.

2. Ordering of ruin probabilities

In this section, we generalize the results of ordering in Tsai (2009) for pairs of ruin probabilities between $\psi_{t,X}$ and $\psi_{t,Y}$, and ψ_X and ψ_Y, valid up to some positive initial surplus. These paired ruin probabilities are from the individual claim size random variables X and Y, respectively, for two continuous time surplus processes perturbed by diffusion with different premium rates, relative security loadings, and variance parameters of the diffusion processes. The results can be used to construct a smooth upper (or lower) bound in the next section.

Tsai (2009) proposed the following conditions for ordering pairs of ruin probabilities:

- Condition 1: $X \leq_{mrl} Y$, or $e_X(t) \leq e_Y(t)$ for all $t \geq 0$.
- Condition 2: $c_X/D_X \geq c_Y/D_Y$.
- Condition 3: $\theta_X \geq \theta_Y$ or $\theta_X/(1+\theta_X) \geq \theta_Y/(1+\theta_Y)$.

It proved that when Conditions 1, 2, and 3 hold then $L_X^\star \leq_{st} L_Y^\star$ and $L_X \leq_{st} L_Y$, or equivalently, $\overline{K}_X(u) \leq \overline{K}_Y(u)$ and $\psi_{t,X}(u) \leq \psi_{t,Y}(u)$ for all $u \geq 0$. As mentioned in Section 1, Condition 1 does not hold in general for all $u \geq 0$ when X and Y come from different distribution families. To study this ordering problem more generally, we relax Condition 1 to $X \leq_{hmrl\ [0,u_*]} Y$, or equivalently, $\int_0^u [1/e_X(s)]ds \geq \int_0^u [1/e_Y(s)]ds,\ u \in [0, u_*]$, for some $u_* \geq 0$.

Lemma 2.1 *$X \leq_{hmrl\ [0,u_*]} Y$ implies $L_c^X \leq_{st\ [0,u_*]} L_c^Y$, or equivalently, $\overline{\Gamma}_X(u) \leq \overline{\Gamma}_Y(u)$ for $u \in [0, u_*]$.*

Proof. First, for $t \geq 0$,

$$\frac{1}{e_Z(t)} = \frac{\overline{F}_Z(t)}{\int_t^\infty \overline{F}_Z(u)du} = -\frac{d}{dt}\ln\left(\int_t^\infty \overline{F}_Z(u)du\right)$$

implies that $\int_t^\infty \overline{F}_Z(u)du = ke^{-\int_0^t [1/e_Z(s)]ds}$, where $k = \int_0^\infty \overline{F}_Z(u)du = E(Z)$ and $Z = X, Y$. Then we have,

$$\frac{\overline{\Gamma}_X(u)}{\overline{\Gamma}_Y(u)} = \frac{\int_u^{\infty} \overline{F}_X(s)ds/E(X)}{\int_u^{\infty} \overline{F}_Y(s)ds/E(Y)} = e^{-\int_0^u [1/e_X(s) - 1/e_Y(s)]ds}, \quad u \geq 0.$$

Therefore, $\int_0^u [1/e_X(s) - 1/e_Y(s)]ds \geq 0$ for $u \in [0, u_*]$ implies $\overline{\Gamma}_X(u) \leq \overline{\Gamma}_Y(u)$ for $u \in [0, u_*]$. Because L_c^Z has the distribution function $\Gamma_Z(Z = X, Y)$, we conclude that $X \leq_{hmrl\ [0,u_*]} Y$ gives $L_c^X \leq_{st\ [0,u_*]} L_c^Y$ by Definitions 4 and 5 in Appendix A. □

Assume that u_c is the smallest positive crossing point such that $e_X(t) \leq e_Y(t)$ for $t \in [0,u_c]$. In this case, we also have $\overline{\Gamma}_X(u) \leq \overline{\Gamma}_Y(u)$ for $t \in [0,u_c]$, that is, $X \leq_{mrl\ [0,u_c]} Y$ implies $X \leq_{hmrl\ [0,u_c]} Y$ which leads to $L_c^X \leq_{st\ [0,u_c]} L_c^Y$. Lemma 1 of Tsai (2009) is just a special case for $u_c = \infty$. Let $u_s = \sup\{u > 0: \int_0^u [1/e_X(s) - 1/e_Y(s)]ds \geq 0\}$. Obviously, $u_c \leq u_s$ and u_s can possibly be obtained by numerical methods, while u_c has a formulaic expression in some cases. Moreover, $X \leq_{hmrl\ [0,u_s]} Y$ implies $L_c^X \leq_{st\ [0,u_s]} L_c^Y$. Therefore, both u_c and u_s can be the choices of u_* in Lemma 2.1. The following examples show the existence of u_*.

Example 1 *Exponential and mixture of two exponentials.*

Let X follow a mixture of two exponential distributions and Y follow the exponential distribution, with density functions and the mean residual lifetime functions

$$f_X(t) = q_1\ \alpha_1\ e^{-\alpha_1 t} + q_2\ \alpha_2\ e^{-\alpha_2 t}, \quad t \geq 0, \tag{2.1}$$

$$e_X(t) = \frac{\frac{q_1}{\alpha_1}\ e^{-\alpha_1 t} + \frac{q_2}{\alpha_2}\ e^{-\alpha_2 t}}{q_1\ e^{-\alpha_1 t} + q_2\ e^{-\alpha_2 t}}, \quad t \geq 0, \tag{2.2}$$

where $0 \leq q_1, q_2 \leq 1$ and $q_1 + q_2 = 1$, and $f_Y(t) = \beta e^{-\beta t}$ and $e_Y(t) = 1/\beta$, $t \geq 0$. By Example 6 in Appendix A, F_X is the decreasing failure rate (DFR) and thus increasing mean residual lifetime (IMRL). Therefore, $e_X(t)$ increases in t from $e_X(0) = q_1/\alpha_1 + q_2/\alpha_2$ *to* $e_X(\infty) = 1/min(\alpha_1,\alpha_2)$. That is, $e_X(t) \in [q_1/\alpha_1 + q_2/\alpha_2, 1/min(\alpha_1,\alpha_2))$.

Case 1: if $E[X] = q_1/\alpha_1 + q_2/\alpha_2 \geq 1/\beta = E[Y]$, then $e_X(t) \geq e_Y(t)$ for $t \geq 0$ and thus $\overline{\Gamma}_X(u) \geq \overline{\Gamma}_Y(u)$ for $u \geq 0$.

Case 2: if $e_X(\infty) = 1/\min(\alpha_1,\alpha_2) \leq 1/\beta = E[Y]$, then $e_X(t) \leq e_Y(t)$ for $t \geq 0$, and hence $\overline{\Gamma}_X(u) \leq \overline{\Gamma}_Y(u)$ for $u \geq 0$.

Case 3: if $q_1/\alpha_1 + q_2/\alpha_2 \leq e_Y(t) = 1/\beta < 1/\min(\alpha_1,\alpha_2)$, then there exists a unique crossing point u_c such that $e_X(t) \leq e_Y(t)$ for $t \in [0,u_c]$ and $e_X(t) \geq e_Y(t)$ for $t \in [u_c,\infty)$. By $e_X(u_c) = e_Y(u_c)$, the crossing point u_c is the solution to equation

$$\frac{q_1}{\alpha_1} e^{-\alpha_1 t} + \frac{q_2}{\alpha_2} e^{-\alpha_2 t} = \frac{q_1}{\beta} e^{-\alpha_1 t} + \frac{q_2}{\beta} e^{-\alpha_2 t},$$

which yields

$$u_c = \frac{\ln[q_1\alpha_2(\beta - \alpha_1)] - \ln[q_2\alpha_1(\alpha_2 - \beta)]}{\alpha_1 - \alpha_2}. \tag{2.3}$$

To find u_s, rewrite $1/e_X(t)$ as

$$\frac{1}{e_X(t)}=\frac{\alpha_1}{\alpha_2-\alpha_1}\frac{\frac{q_1}{\alpha_1}(\alpha_2-\alpha_1)\,e^{(\alpha_2-\alpha_1)t}}{\frac{q_1}{\alpha_1}e^{(\alpha_2-\alpha_1)t}+\frac{q_2}{\alpha_2}}+\frac{\alpha_2}{\alpha_1-\alpha_2}\frac{\frac{q_2}{\alpha_2}(\alpha_1-\alpha_2)\,e^{(\alpha_1-\alpha_2)t}}{\frac{q_2}{\alpha_2}e^{(\alpha_1-\alpha_2)t}+\frac{q_1}{\alpha_1}}.$$

Then after straightforward integrations, we get

$$\int_0^u\left[\frac{1}{e_X(t)}-\frac{1}{e_Y(t)}\right]dt=\ln\left[\frac{\frac{q_1}{\alpha_1}+\frac{q_2}{\alpha_2}}{\frac{q_1}{\alpha_1}e^{-\alpha_1 u}+\frac{q_2}{\alpha_2}e^{-\alpha_2 u}}\right]-\beta u. \tag{2.4}$$

So, $\int_0^u[1/e_X(t)-1/e_Y(t)]dt\geq 0$ if and only if $(q_1/\alpha_1)\,e^{-\alpha_1 u}+(q_2/\alpha_2)\,e^{-\alpha_2 u}\leq[q_1/\alpha_1+q_2/\alpha_2]\,e^{-\beta u}$, and u_s is the smallest positive root to $(q_1/\alpha_1)e^{-\alpha_1 u}+(q_2/\alpha_2)e^{-\alpha_2 u}=[q_1/\alpha_1+q_2/\alpha_2]e^{-\beta u}$. With $q_1=0.4$, $q_2=0.6$, $\alpha_1=2$, $\alpha_2=0.75$, and $\beta=0.8$, we have $1=q_1/\alpha_1+q_2/\alpha_2\leq e_Y(t)=1/\beta=1.25\leq 1/\min(\alpha_1,\alpha_2)=4/3$; u_c can be obtained by Eq. (2.3) as $u_c=\ln(6)/1.25=1.433408$, and $u_s\approx 4.443529$ by the numerical evaluation.

Example 2 *Pareto and exponential distributions.*

Let X ~ Pareto(α,τ) with $\alpha>1$ and Y ~ Exp(β) with $\beta>0$. Then for $t\geq 0$,

$$f_X(t)=\frac{\alpha\tau^{\alpha}}{(t+\tau)^{\alpha+1}},\qquad e_X(t)=\frac{t+\tau}{\alpha-1},$$
$$f_Y(t)=\beta\,e^{-\beta t},\qquad e_Y(t)=1/\beta.$$

Note that $e_X(t)$ increases in t from the minimum value $e_X(0)=E[X]=\tau/(\alpha-1)$ to infinity.

Case 1: if $E[X]=\tau/(\alpha-1)\geq 1/\beta=E[Y]$, then $e_X(t)\geq e_Y(t)$ *for* $t\geq 0$ and thus $\overline{\Gamma}_X(u)\geq\overline{\Gamma}_Y(u)$ for $u\geq 0$.

Case 2: if $E[X]=\tau/(\alpha-1)<1/\beta=E[Y]$, then the unique crossing point $u_c=[(\alpha-1)/\beta]-\tau>0$ and $\overline{\Gamma}_X(u)\leq\overline{\Gamma}_Y(u)$ for $u\leq u_c$. To find u_s, let

$$\int_0^u\left[\frac{1}{e_X(t)}-\frac{1}{e_Y(t)}\right]dt=(\alpha-1)\ln\left(1+\frac{u}{\tau}\right)-\beta u\geq 0;$$

then u_s is the smallest positive root of $u/\tau+1=e^{[\beta/(\alpha-1)]u}$. When $\tau=0.5$, $\alpha=2$, and $\beta=1$, it can be obtained that $u_c=0.5$ and $u_s\approx 1.256431$. When $\tau=6$, $\alpha=7$, and $\beta=0.5$, we get $u_c=6$ and $u_s\approx 15.077175$.

Example 3 *Two Pareto distributions.*

Let X ~ Pareto(α_1,τ_1) with $e_X(t)=(t+\tau_1)/(\alpha_1-1)$, $t\geq 0,\alpha_1>1$, and Y ~ Pareto(α_2,τ_2) with $e_Y(t)=(t+\tau_2)/(\alpha_2-1), t\geq 0$, $\alpha_2>1$. Both $e_X(t)$ and $e_Y(t)$ linearly increase from $e_X(0)$ and $e_Y(0)$, respectively, to infinity.

Case 1: if $e_X(0)\leq e_Y(0)$ and $\alpha_1\geq\alpha_2$, then $e_X(t)\leq e_Y(t)$ *and* $\overline{\Gamma}_X(u)\leq\overline{\Gamma}_Y(u)$ *for* $u\geq 0$.

Case 2: if $e_X(0) < e_Y(0)$ and $\alpha_1 < \alpha_2$, then there exists a unique crossing point u_c, such that $e_X(t) \le e_Y(t)$ for $t \in [0, u_c]$ and $e_X(t) \ge e_Y(t)$ for $t \in [u_c, \infty)$ where $u_c = [\tau_2(\alpha_1 - 1) - \tau_1(\alpha_2 - 1)]/(\alpha_2 - \alpha_1)$ is the root of $e_X(t) = e_Y(t)$. To find u_s, let

$$\int_0^u \left[\frac{1}{e_X(t)} - \frac{1}{e_Y(t)}\right] dt = (\alpha_1 - 1)\ln\left(1 + \frac{u}{\tau_1}\right) - (\alpha_2 - 1)\ln\left(1 + \frac{u}{\tau_2}\right) \ge 0.$$

Then u_s is the smallest positive root of $(1 + u/\tau_1)^{\alpha_1 - 1} = (1 + u/\tau_2)^{\alpha_2 - 1}$. When $\tau_1 = 1$, $\alpha_1 = 2$, and $\tau_2 = \alpha_2 = 3$, we have $u_c = 1$ and $u_s = 3$.

We present below more properties for the ordering of two random variables over a finite interval, followed by the main result to be used for constructing the bounds of the ruin probability in the next section.

Lemma 2.2 *If $X_1 \le_{st\ [0,u_1]} Y_1$ and $X_2 \le_{st\ [0,u_2]} Y_2$, then $X_1 + X_2 \le_{st\ [0, min(u_1, u_2)]} Y_1 + Y_2$.*

Proof: For $i = 1,2$, $X_i \le_{st\ [0,u_i]} Y_i$ if and only if $\overline{F}_{X_i}(u) \le \overline{F}_{Y_i}(u)$ which is equivalent to $F_{X_i}(u) \ge F_{Y_i}(u), 0 \le u \le u_i$. Let $u_3 = \min(u_1, u_2)$; then $F_{X_1+X_2}(u) = \int_0^u F_{X_1}(u-t)dF_{X_2}(t) \ge \int_0^u F_{Y_1}(u-t)dF_{X_2}(t) = \int_0^u F_{X_2}(u-t)dF_{Y_1}(t) \ge \int_0^u F_{Y_2}(u-t)dF_{Y_1}(t) = \int_0^u F_{Y_1}(u-t)dF_{Y_2}(t) = F_{Y_1+Y_2}(u)$ for $0 \le u \le u_3$, that is, $X_1 + X_2 \le_{st\ [0,u_3]} Y_1 + Y_2$. □

Lemma 2.3 *If $X \le_{hmrl\ [0,u_*]} Y$ and $c_X/D_X \ge c_Y/D_Y$, then $L_c^X + L_o^X \le_{st\ [0,u_*]} L_c^Y + L_o^Y$, or equivalently, $\overline{G}_X(u) \le \overline{G}_Y(u)$ for all $0 \le u \le u_*$.*

Proof: If $X \le_{hmrl\ [0,u_*]} Y$, then $L_c^X \le_{st\ [0,u_*]} L_c^Y$ by Lemma 2.1. If $c_X/D_X \ge c_Y/D_Y$, then $\overline{H}_X(u) = e^{-(c_X/D_X)u} \le e^{-(c_Y/D_Y)u} = \overline{H}_Y(u)$ for all $u \ge 0$, implying $L_o^X \le_{st\ [0,\infty)} L_o^Y$ since L_o^Z has the distribution function $H_Z\ (Z = X, Y)$. Now by Lemma 2.2, $L_c^X + L_o^X \le_{st\ [0,u_*]} L_c^Y + L_o^Y$. □

Theorem 2.1 *If $X \le_{hmrl\ [0,u_*]} Y$, $c_X/D_X \ge c_Y/D_Y$, and $\theta_X \ge \theta_Y$, then*

(a) $L_X^\star \le_{st\ [0,u_*]} L_Y^\star$ (or equivalently, $\overline{K}_X(u) \le \overline{K}_Y(u)$ for all $u \in [0, u_*]$),
(b) $L_X \le_{st\ [0,u_*]} L_Y$ (or equivalently, $\psi_{t,X}(u) \le \psi_{t,Y}(u)$ for all $u \in [0, u_*]$), and
(c) $\psi_{d,X} \le_{sl\ [0,u_*]} \psi_{d,Y}$.

Proof: First by Lemma 2.3, $L_c^X + L_o^X \le_{st\ [0,u_*]} L_c^Y + L_o^Y$, which implies, by Lemma 2.2, that

$$\sum_{n=1}^{m} (L_{o,n-1}^X + L_{c,n}^X) \le_{st\ [0,u_*]} \sum_{n=1}^{m} (L_{o,n-1}^Y + L_{c,n}^Y), \quad m = 1, 2, \ldots,$$

or equivalently, for all $u \in [0, u_*]$,

$$\Pr\left[\sum_{n=1}^{m} (L_{o,n-1}^X + L_{c,n}^X) \le u\right] \ge \Pr\left[\sum_{n=1}^{m} (L_{o,n-1}^Y + L_{c,n}^Y) \le u\right], \quad m = 1, 2, \ldots \qquad (2.5)$$

Then by Eq. (2.5), we have (see Eq. (1.2) for the definition of $L_X^\star$)

$$F_{L_X^\star}(u) = \sum_{m=0}^{\infty} \Pr\left[\sum_{n=1}^{N_X} (L_{o,n-1}^X + L_{c,n}^X) \le u \mid N_X = m\right] \Pr[N_X = m]$$
$$= \sum_{m=0}^{\infty} \Pr\left[\sum_{n=1}^{m} (L_{o,n-1}^X + L_{c,n}^X) \le u\right] \Pr[N_X = m]$$
$$\ge \sum_{m=0}^{\infty} \Pr\left[\sum_{n=1}^{m} (L_{o,n-1}^Y + L_{c,n}^Y) \le u\right] \Pr[N_X = m]$$
$$= \sum_{m=0}^{\infty} \Pr\left[\sum_{n=1}^{N_X} (L_{o,n-1}^Y + L_{c,n}^Y) \le u \mid N_X = m\right] \Pr[N_X = m]$$
$$= E_{N_X}\left[F_{\sum_{n=1}^{N_X} (L_{o,n-1}^Y + L_{c,n}^Y)|N_X}(u)\right].$$

Since N_X and N_Y are geometrically distributed with means $1/\theta_X$ and $1/\theta_Y$, respectively, and $1/\theta_X \le 1/\theta_Y$, it follows $N_X \le_{st} N_Y$ by Example 6 in Tsai (2009). That is, for any non-increasing real function w on $[0,\infty)$, $E_{N_X}[w(N_X)] \ge E_{N_Y}[w(N_Y)]$ by Definition 5 in Appendix A. Define

$$w_u(M = m) \overset{\Delta}{=} F_{\sum_{n=1}^{M} (L_{o,n-1}^Y + L_{c,n}^Y)|M=m}(u) = \Pr\left[\sum_{n=1}^{m} (L_{o,n-1}^Y + L_{c,n}^Y) \le u\right].$$

If we can prove that, for each fixed $u \in [0, u_*]$, $w_u(M = m)$ is non-increasing in m, then

$$F_{L_X^\star}(u) \ge E_{N_X}\left[F_{\sum_{n=1}^{N_X} (L_{o,n-1}^Y + L_{c,n}^Y)|N_X}(u)\right] \ge E_{N_Y}\left[F_{\sum_{n=1}^{N_Y} (L_{o,n-1}^Y + L_{c,n}^Y)|N_Y}(u)\right] = F_{L_Y^\star}(u),$$

that is, $L_X^\star \le_{st\ [0,u_*]} L_Y^\star$, or $\overline{K}_X(u) = \overline{F}_{L_X^\star}(u) \le \overline{F}_{L_Y^\star}(u) = \overline{K}_Y(u)$ for $u \in [0, u_*]$.

Now, $\Sigma_{n=1}^{m}(L_{o,n-1}^Y + L_{c,n}^Y) \le u$ implies $\Sigma_{n=1}^{m^*}(L_{o,n-1}^Y + L_{c,n}^Y) \le u$ for all $m^* \le m$, yielding $\Pr[\Sigma_{n=1}^{m}(L_{o,n-1}^Y + L_{c,n}^Y) \le u] \le \Pr[\Sigma_{n=1}^{m^*}(L_{o,n-1}^Y + L_{c,n}^Y) \le u]$ for all $m^* \le m$, that is, $w_u(M = m)$ is non-increasing in m for each fixed $u \in [0, u_*]$. Thus (a) is proved.

Because $L_o^X \le_{st\ [0,\infty)} L_o^Y$ and $L_X^\star \le_{st\ [0,u_*]} L_Y^\star$, we get $L_X = L_X^\star + L_o^X \le_{st\ [0,u_*]} L_Y^\star + L_o^Y = L_Y$ by Lemma 2.2. Equivalently, $\psi_{t,X}(u) = \Pr(L_X > u) \le \Pr(L_Y > u) = \psi_{t,Y}(u)$ for $0 \le u \le u_*$, which proves (b).

Finally, by Eq. (2.12) in Tsai (2003), $-d\psi_t(u)/du = (c/D)[\theta/(1+\theta)]\psi_d(u)$, the stop-loss transform of ψ_d (see Definition 7 in Appendix A) has the following expression:

$$\prod_{\psi_d}(u) \overset{\Delta}{=} -\int_0^\infty (s-u)d\psi_d(s) = \int_0^\infty \psi_d(s)ds = -\frac{D}{c}\frac{1+\theta}{\theta}\int_0^\infty d\psi_t(s) = \frac{D}{c}\frac{1+\theta}{\theta}\psi_t(u).$$

As $[c_X/D_X][\theta_X/(1+\theta_X)] \ge [c_Y/D_Y][\theta_Y/(1+\theta_Y)]$ is implied by $c_X/D_X \ge c_Y/D_Y$ and $\theta_X \ge \theta_Y$, it turns out that $\Pi_{\psi_{d,X}}(u) = [D_X/c_X][(1+\theta_X)/\theta_X]\psi_{t,X}(u) \le [D_Y/c_Y][(1+\theta_Y)/\theta_Y]\psi_{t,Y}(u) = \Pi_{\psi_{d,Y}}(u)$ for $u \in [0, u_*]$. Thus (c) is concluded by Definition 7 in Appendix A. □

When the diffusion component is removed ($\sigma_Z = 0$), the ruin probability $\psi_Z(u)$ expressed as a compound geometric tail distribution in Eq. (1.6) does not contain H_Z (the distribution of L_o^Z). Therefore, the condition $c_X/D_X \ge c_Y/D_Y$ in Theorem 2.1 for $L_o^X \le_{st\ [0,\infty)} L_o^Y$ is removed to form the following result.

Corollary 2.1 *For surplus process (1.5), if* $X \leq_{hmrl\ [0,u_*]} Y$ *and* $\theta_X \geq \theta_Y$, *then* $L^\bullet_X \leq_{st\ [0,u_*]} L^\bullet_Y$ *or* $\psi_X(u) \leq \psi_Y(u)$ *for all* $u \in [0, u_*]$.

3. Ruin probability bounds from exponential claim amounts

In this section, we first review several existing upper bounds for ψ_X, which are established based on the fact that ψ_X can be expressed as the tail distribution of a compound geometric random variable. Since $\overline{K}_X$ can also be written as the tail distribution of a compound geometric random variable, these bounds are then being modified to the ones for $\overline{K}_X$ to include the diffusion component for the sake of comparison. Note that from Eq. (1.4), we have

$$\psi_{t,X}(u) = \overline{K_X * H_X}(u) = \overline{H}_X(u) + \int_0^u \overline{K}_X(u-t)dH_X(t), \quad u \geq 0 \ ;$$

thus the upper bound for $\psi_{t,X}(u)$ can be further obtained from any given upper bound $UB(u)$ for $\overline{K}_X(u)$, given by

$$\psi_{t,X}(u) \leq \overline{H}_X(u) + \int_0^u UB(u-t)dH_X(t), \tag{3.1}$$

where $\overline{H}_X(u) = e^{-(c_X/D_X)u}$, and the bound holds for the same interval range as $UB(u)$.

Our main objective of this section is to propose an effective algorithm for constructing a smooth upper (lower) bound obtained from ruin probabilities with exponential claims for ψ_X, provided that the mean residual lifetime function $e_X(t)$ of X is non-decreasing (non-increasing) in $t \in [0, u_*]$ for certain positive u_*, by using the ordering result from Corollary 2.1. The corresponding one for $\psi_{t,X}$, supported by Theorem 2.1, can always be obtained as explained above. Note that the constructed upper (lower) bound is valid up to a certain positive value. Finally, two numerical examples are given for illustration when the underlying distributions are Pareto and a mixture of two exponentials, respectively; our proposed bound is compared with these existing bounds.

3.1. Some existing upper bounds for $\psi_X(u)$

The most famous upper bound for $\psi_X(u)$ is the so-called Lundberg's bound given by

$$\psi_X(u) \leq e^{-\kappa_0 u} \stackrel{\Delta}{=} UB_{L_0}(u), \quad u \geq 0, \tag{3.2}$$

where $\kappa_0 > 0$ is the smallest positive root to a Lundberg's fundamental equation

$$\lambda_X + c_X s = \lambda_X \int_0^u e^{su} dF_X(u) \tag{3.3}$$

(see Klugman *et al.* (2004)). Moreover, Broeckx *et al.* (1986), Willmot (1994), and Cai & Garrido (1998) proposed the following upper bounds for $\psi_X(u)$, respectively,

$$\psi_X(u) \leq \frac{\overline{\Gamma}_X(u)u + \int_0^u t d\Gamma_X(t)}{[\overline{\Gamma}_X(u) + \theta_X]u + \int_0^u t d\Gamma_X(t)} = \frac{\int_0^u \overline{\Gamma}_X(t)dt}{\int_0^u \overline{\Gamma}_X(t)dt + \theta_X u} \stackrel{\Delta}{=} UB_{B_0}(u), \quad u > 0, \tag{3.4}$$

$$\psi_X(u) \le \frac{E[L_X^{\bullet}]}{E[L_X^{\bullet}] + u} = \frac{\int_0^\infty \overline{\Gamma}_X(t)dt}{\int_0^\infty \overline{\Gamma}_X(t)dt + \theta_X u} \triangleq UB_{W_0}(u), \quad u \ge 0, \tag{3.5}$$

and

$$\begin{aligned}\psi_X(u) &\le \frac{E[L_X^{\bullet}]\Gamma_X(u) + u\,\overline{\Gamma}_X(u)}{E[L_X^{\bullet}]\Gamma_X(u) + [1 + \theta_X]u} \\ &= \frac{\left[\int_0^\infty \overline{\Gamma}_X(t)dt\right]\Gamma_X(u) + \theta_X\, u\,\overline{\Gamma}_X(u)}{\left[\int_0^\infty \overline{\Gamma}_X(t)dt\right]\Gamma_X(u) + \theta_X[1 + \theta_X]u} \triangleq UB_{C_0}(u), \quad u > 0. \end{aligned} \tag{3.6}$$

In the inequalities above, we have used that $E[L_X^{\bullet}] = E[N_X]E[L_c^X] = (1/\theta_X)\int_0^\infty \overline{\Gamma}_X(t)dt$. Obviously, $UB_{B_0}(u) \le UB_{W_0}(u)$ for all $u > 0$, and $UB_{B_0}(u) \approx UB_{W_0}(u)$ for large u. Cai & Garrido (1998) also showed that UB_{C_0} is tighter than UB_{W_0} (that is, $UB_{C_0}(u) \le UB_{W_0}(u)$ for all $u > 0$).

3.2. Upper bounds for $\overline{K}_X(u)$

For $\sigma_X > 0$, by the fact that both Eqs. (1.3) for $\overline{K}_X(u)$ and (1.6) for $\psi_X(u)$ have the compound geometric tail distribution expressions, inequality in Eq. (3.2) can be easily modified to

$$\overline{K}_X(u) \le e^{-\kappa u} \triangleq UB_L(u), \quad u \ge 0, \tag{3.7}$$

where $\kappa > 0$ is the smallest positive root to a generalized Lundberg's equation

$$\lambda_X + c_X s - D_X s^2 = \lambda_X \int_0^\infty e^{su} dF_X(u). \tag{3.8}$$

Analogous inequalities of Eqs. (3.4)–(3.6) for $\overline{K}_X(u)$ are of the following expressions:

$$\overline{K}_X(u) \le \frac{\overline{G}_X(u)u + \int_0^u t dG_X(t)}{[\overline{G}_X(u) + \theta_X]u + \int_0^u t dG_X(t)} = \frac{\int_0^u \overline{G}_X(t)dt}{\int_0^u \overline{G}_X(t)dt + \theta_X u} \triangleq UB_B(u), \quad u \ge 0,$$

$$\overline{K}_X(u) \le \frac{E[L_X^{\star}]}{E[L_X^{\star}] + u} = \frac{\int_0^\infty \overline{G}_X(t)dt}{\int_0^\infty \overline{G}_X(t)dt + \theta_X u} \triangleq UB_W(u), \quad u \ge 0,$$

and

$$\begin{aligned}\overline{K}_X(u) &\le \frac{E[L_X^{\star}]G_X(u) + u\,\overline{G}_X(u)}{E[L_X^{\star}]G_X(u) + [1 + \theta_X]u} \\ &= \frac{\left[\int_0^\infty \overline{G}_X(t)dt\right]G_X(u) + \theta_X\, u\,\overline{G}_X(u)}{\left[\int_0^\infty \overline{G}_X(t)dt\right]G_X(u) + \theta_X[1 + \theta_X]u} \triangleq UB_C(u), \quad u \ge 0.\end{aligned}$$

Note that expressions of UB_{L_0} and UB_L are very simple. However, κ_0 and κ in Eqs. (3.2) and (3.7) solved from the corresponding Lundberg's equations (3.3) and (3.8) need the help of numerical methods, and more importantly they do not always exist for heavy-tailed (for example, Pareto and Lognormal) and medium-tailed (such as inverse Gaussian and generalized inverse Gaussian) claim size distributions. Details on related topics can be found in Teugels & Sundt (2004).

3.3. Bounds from exponential claims

Motivated by the fact that the ruin probability of surplus process (1.5) can be obtained explicitly when the claims are exponentially distributed, we let X and Y be the underlying and exponential random variables, respectively, with $E[Y]=e_Y(t)=1/\beta$ where β is going to be determined. If $e_X(u)$ is non-decreasing in $u\in[0,u_*]$ for some $u_*>0$, then for any fixed $u\in[0,u_*]$, setting equation

$$\int_0^u\left[\frac{1}{e_X(t)}-\frac{1}{e_Y(t)}\right]dt=\int_0^u\left[\frac{1}{e_X(t)}-\beta\right]dt=0$$

yields

$$\beta=\frac{1}{u}\int_0^u\frac{1}{e_X(t)}dt\overset{\Delta}{=}\beta(u),\quad u\in[0,u_*], \tag{3.9}$$

implying $\int_0^s[1/e_X(t)-1/e_Y(t)]dt=\int_0^s[1/e_X(t)-\beta(u)]dt\geq 0$ for $s\in[0,u]$, or $X\leq_{hmrl\,[0,u]}Y$ (see Figure 2). Now, if we choose random variable Y to be exponentially distributed with mean $1/\beta(u)$, where $\beta(u)$ is given by Eq. (3.9), and let $\theta_Y=\theta_X$, then by Corollary 2.1, $\psi_X(s)\leq\psi_Y(s)$ for all $s\in[0,u]$; the explicit expression of $\psi_Y(s)$ is thus an upper bound of $\psi_X(s)$, the ruin probability of X.

Furthermore, it is easy to check that

$$\frac{d\beta(u)}{du}=\frac{1}{u^2}\left[\frac{u}{e_X(u)}-\int_0^u\frac{1}{e_X(t)}dt\right]\leq 0,\quad u\in[0,u_*],$$

as $1/e_X(t)$ is assumed non-increasing in $t\in[0,u_*]$, implying that $\beta(u)$ is non-increasing in $u\in[0,u_*]$, with $\beta(0)=1/e_X(0)=1/E[X]$. Now consider the interval $[0,u_2]$ where $u_2\leq u_*$, and let $0\leq u_1<u_2$; then $\beta(u_1)>\beta(u_2)$. If Y_1 and Y_2 are two exponentially distributed random variables with means $1/\beta(u_1)$ and $1/\beta(u_2)$, respectively, then we have $X\leq_{hmrl\,[0,u_1]}Y_1$, $X\leq_{hmrl\,[0,u_2]}Y_2$, as well as $Y_1\leq_{hmrl\,[0,\infty)}Y_2$. In addition, let $\theta_{Y_1}=\theta_{Y_2}=\theta_X$. It turns out by Corollary 2.1 that the following results hold: $\psi_X(s)\leq\psi_{Y_1}(s)\leq\psi_{Y_2}(s)$ for all $s\in[0,u_1]$, and $\psi_X(s)\leq\psi_{Y_2}(s)$ for all $s\in[0,u_2]$. Apparently, the upper bound resulting from Y_1 is tighter than the one from Y_2 in the interval $[0,u_1]$. As a result, $\psi_{Y_1}(s)$, $s\in[0,u_1]$ and $\psi_{Y_2}(s)$, $s\in[u_1,u_2]$, is a tighter upper bound than the sole $\psi_{Y_2}(s)$ for $\psi_X(s)$, $s\in[0,u_2]$. Alternatively, as the ruin probability is a convex function, we can take $\psi_{Y_1}(s)$, $s\in[0,u_1]$, and a segment connecting $(u_1,\psi_{Y_1}(u_1))$ and $(u_2,\psi_{Y_2}(u_2))$ to get an upper bound. The idea is used to construct a piecewise linear upper bound resulting from a series of exponentially distributed random variables over the equally partitioned interval $[0, u_*]$. We provide a step-by-step and effective algorithm for constructing such an upper bound for ψ_X in the following.

ASSUMPTION: $e_X(u)$ is non-decreasing in $u \in [0, u_*]$ for some $u_* > 0$, and value of θ_X is given.

Algorithm: The following steps are for constructing a piecewise upper bound for ψ_X:

i. Divide $[0, u_*]$ into n subintervals of equal width, $[u_k, u_{k+1}]$, $k = 0, \ldots, n-1$, where $u_k = k\, u_*/n,\ \ k = 0, 1, \ldots, n$ with $u_0 = 0$ and $u_n = u_*$.

ii. For each of $k = n, (n-1), \ldots, 1, 0$,
   (1) let $Y_k \sim \text{Exp}(\beta_k)$ where $\beta_k = (1/u_k)\int_0^{u_k} [1/e_X(t)]dt$ obtained from Eq. (3.9), implying that $X \leq_{hmrl\ [0,u_k]} Y_k$, and
   (2) let $\theta_{Y_k} = \theta_X$.
   Both (1) and (2) ensure that $\psi_X(u) \leq \psi_{Y_k}(u)$ for $u \in [0, u_k]$ by Corollary 2.1.

iii. Connect $(u_{k-1}, \psi_{Y_{k-1}}(u_{k-1}))$ and $(u_k, \psi_{Y_k}(u_k))$ by a linear segment (see Figure 1),

$$\frac{\psi_{Y_k}(u_k) - \psi_{Y_{k-1}}(u_{k-1})}{u_k - u_{k-1}}(u - u_{k-1}) + \psi_{Y_{k-1}}(u_{k-1}), \quad u \in [u_{k-1}, u_k],$$

for $k = 1, \ldots, n$ to form a piecewise linear upper bound where $\psi_{Y_k}(u)$ is known as:

$$\psi_{Y_k}(u) = \frac{1}{1+\theta_{Y_k}} e^{-\frac{\theta_{Y_k}}{1+\theta_{Y_k}}\beta_k u} = \frac{1}{1+\theta_X} e^{-\frac{\theta_X}{1+\theta_X}\beta_k u}, \quad u \in [0, u_k]. \tag{3.10}$$

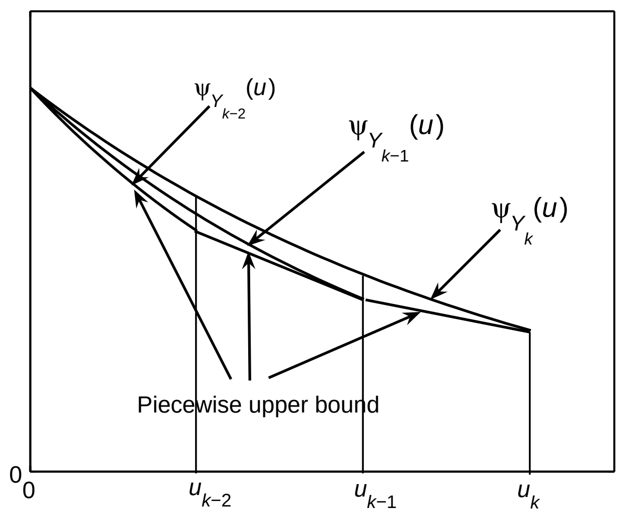

Figure 1. A piecewise linear upper bound.

Finally, since each of the segments is obtained by connecting two adjacent right end points, as n (the number of subintervals) goes to infinity, the constructed piecewise linear upper bound converges to a smooth curve which is our desired upper bound denoted by UB_{T_0}. That is,

$$UB_{T_0}(u) = \frac{1}{1+\theta_X} e^{-\frac{\theta_X}{1+\theta_X}\beta(u)u}, \quad u \in [0, u_*], \tag{3.11}$$

where $\beta(u)$ is given in Eq. (3.9).

Two of the main advantages of using exponential distribution for constructing bounds are that exponential distribution has a constant mean residual lifetime function and the ruin probability resulting from the exponential distribution has a formula-based expression. In Eq. (3.11), the only parameter $\beta(u)$ of the exponential distribution can be easily determined from Eq. (3.9). For other random variable Y, we might have to obtain the parameter(s) from $\int_0^u [1/e_X(t) - 1/e_Y(t)]dt = 0$ by some numerical method, and the parameter(s) may not be unique.

Note that the case where $e_X(u)$ is non-increasing in $u \in [0, u_*]$, for some $u_* > 0$, can be considered analogously and the corresponding results are lower bounds of $\psi_X(u)$ for $u \in [0, u_*]$. Other remarks on the algorithm are as follows.

Remarks

(1) When X is also exponentially distributed with mean $1/\beta_X$, Eq. (3.9) leads to $\beta(u) = \beta_X$, implying $UB_{T_0}(u) = \psi_X(u)$ for $u \in [0, u_*]$. However, none of $UB_{L_0}(u)$, $UB_{B_0}(u)$, $UB_{W_0}(u)$, and $UB_{C_0}(u)$ given in Section 3.1 completely overlap with $\psi_X(u)$. Moreover, $UB_{T_0}(0) = 1/(1+\theta_X)$ by Eq. (3.11). However, among the upper bounds in Section 3.1, only $UB_{B_0}(0) = 1/(1+\theta_X)$, which coincides with $\psi_X(0) = 1/(1+\theta_X)$. Thus, we argue that our upper bound UB_{T_0} is better than all others mentioned above from the viewpoint of these two aspects.

(2) When the non-decreasing $e_X(t)$ is unbounded above (see Example 3 for the Pareto case), by the result from Tsai (2009) we cannot find any random variable Y such that $e_X(t) \leq e_Y(t)$ for all $t \geq 0$, and there is a formula-based expression (for example, Y is a mixture of exponentials or a mixture of Erlangs) for $\psi_Y(u)$ as an upper bound of $\psi_X(u)$ for all $u \geq 0$. Though this algorithm gives an efficient approach to construct a smooth upper bound with some favorable features only up to a certain u_*, the upper bound can be valid for all $u \geq 0$ provided that $e_X(t)$ is non-increasing for all $t \geq 0$. See Examples 4 and 5.

(3) Recall that $e_X(u) \leq e_Y(u)$ for $u \in [0, u_*]$ implies $\int_0^u [1/e_X(s) - 1/e_Y(s)ds \geq 0$ *for* $u \in [0, u_*]$. The latter is one of the sufficient conditions for Corollary 2.1. There is an alternative value for $\beta(u)$, denoted by $\beta^*(u)$, which produces a looser upper bound than UB_{T_0}. Whenever u is given, let Y* be also exponentially distributed with $E[Y^*] = e_{Y^*}(t) = 1/\beta^*(u)$ where $\beta^*(u) = 1/e_{Y^*}(t) = 1/e_X(u)$, and $UB_{T_0^*}$ be corresponding constructed smooth upper bound. Since $1/e_X(u)$ is assumed non-increasing in u for $u \in [0, u_*]$, we get $\beta(u) = (1/u)\int_0^u [1/e_X(t)]dt \geq 1/e_X(u) = \beta^*(u)$ (see Figure 2),

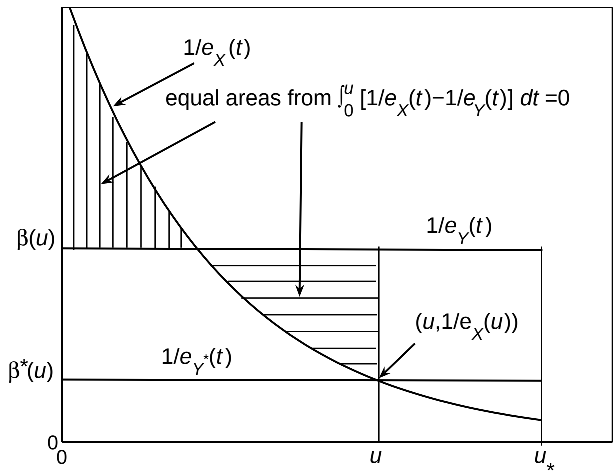

Figure 2. Illustration of $\beta^*(u) \leqq \beta(u)$.

implying $UB_{T_0}(u) \leq UB_{T_0^*}(u)$ for $u \in [0, u_*]$ by (3.11). Though it is easier to calculate $\beta^*(u) = 1/e_X(u)$ than $\beta(u) = (1/u)\int_0^u [1/e_X(t)]dt$, bound $UB_{T_0^*}(u)$ could be much looser than $UB_{T_0}(u)$.

(4) The piecewise linear upper bounds for $\overline{K}_X(u)$ for surplus process (1.1) can also be developed similarly as follows: (ii) of (2) in the algorithm above is modified to 'let $\theta_{Y_k} = \theta_X$ and $c_{Y_k}/D_{Y_k} = c_X/D_X$' (by Theorem 2.1 (*a*), $\overline{K}_X(u) \leq \overline{K}_{Y_k}(u)$ for $u \in [0, u_k]$). Moreover, ψ in (ii) and (iii) of Algorithm is replaced by $\overline{K}$, and Eq. (3.10) is replaced with (see Tsai (2006)) $\overline{K}_{Y_k}(u) = \theta_X(D_1 e^{-s_1 u} + D_2 e^{-s_2 u})$, $u \in [0, u_k]$, where

$$s_1 = \frac{(\frac{c_X}{D_X} + \beta_k) - \sqrt{\left(\frac{c_X}{D_X} - \beta_k\right)^2 + 4\frac{c_X}{D_X}\frac{\beta_k}{1+\theta_X}}}{2},$$

$$s_2 = \frac{(\frac{c_X}{D_X} + \beta_k) + \sqrt{\left(\frac{c_X}{D_X} - \beta_k\right)^2 + 4\frac{c_X}{D_X}\frac{\beta_k}{1+\theta_X}}}{2},$$

$$D_1 = \frac{s_2}{\theta_X(1+\theta_X)\sqrt{\left(\frac{c_X}{D_X} - \beta_k\right)^2 + 4\frac{c_X}{D_X}\frac{\beta_k}{1+\theta_X}}},$$

and

$$D_2 = -\frac{s_1}{\theta_X(1+\theta_X)\sqrt{\left(\frac{c_X}{D_X} - \beta_k\right)^2 + 4\frac{c_X}{D_X}\frac{\beta_k}{1+\theta_X}}}.$$

Note that $\beta_k = (1/u_k)\int_0^{u_k} [1/e_X(t)]dt$ is unchanged. The smooth upper bound denoted by $\mathrm{UB}_T(u)$ is given by

$$UB_T(u) = \theta_X(D_1 e^{-s_1 u} + D_2 e^{-s_2 u}), \quad u \in [0, u_*], \tag{3.12}$$

where s_i and D_i, i = 1,2, are the same as those above with β_k replaced by $\beta(u)$ given in Eq. (3.9). Also, the upper bound for $\psi_{t,X}(u)$ is easily obtained from Eqs. (3.1) and (3.12) as

$$e^{-\frac{c_X}{D_X}u} + \theta_X \frac{c_X}{D_X} \sum_{k=1}^{2} \frac{D_k}{s_k - \frac{c_X}{D_X}} (e^{-\frac{c_X}{D_X}u} - e^{-s_k u}), \quad u \in [0, u_*].$$

The following examples give upper bounds on ruin probabilities for insurance claims from a mixture of two exponentials and from a Pareto distribution, respectively. Since $UB_{B_0}(u) \le UB_{W_0}(u)$, $UB_{C_0}(u) \le UB_{W_0}(u)$, and $UB_{T_0}(u) \le UB_{T_0^*}(u)$, we would not study $UB_{W_0}(u)$ and $UB_{T_0^*}(u)$ (as well as $UB_W(u)$ and $UB_{T^*}(u)$ for the diffusion case) to get a better view of the plotted figures.

Example 4 *Upper bounds for ruin probabilities from a mixture of two exponentials.*

Let X follow a mixture of two exponential distributions with $f_X(t)$ and $e_X(t)$ given by Eqs. (2.1) and (2.2), respectively. It is known that $e_X(t)$ is increasing in t. To get upper bounds UB_{B_0} and UB_{C_0} for ψ_X from Eqs. (3.4) and (3.6), we need $\overline{\Gamma}_X(u)$, $\int_0^u \overline{\Gamma}_X(t)dt$ and $E[L_X^{\bullet}]$ as follows:

$$\overline{\Gamma}_X(u) = \frac{\int_u^\infty \overline{F}_X(t)dt}{E[X]} = \frac{\frac{q_1}{\alpha_1}e^{-\alpha_1 u} + \frac{q_2}{\alpha_2}e^{-\alpha_2 u}}{\frac{q_1}{\alpha_1} + \frac{q_2}{\alpha_2}}, \quad u \ge 0,$$

$$\int_0^u \overline{\Gamma}_X(t)dt = \frac{\frac{q_1}{\alpha_1^2}(1 - e^{-\alpha_1 u}) + \frac{q_2}{\alpha_2^2}(1 - e^{-\alpha_2 u})}{\frac{q_1}{\alpha_1} + \frac{q_2}{\alpha_2}}, \quad u \ge 0,$$

and

$$E[L_X^{\bullet}] = E[N_X]E[L_c^X] = \frac{1}{\theta_X}\int_0^\infty \overline{\Gamma}_X(t)dt = \frac{1}{\theta_X}\frac{\frac{q_1}{\alpha_1^2} + \frac{q_2}{\alpha_2^2}}{\frac{q_1}{\alpha_1} + \frac{q_2}{\alpha_2}}.$$

To get UB_B and UB_C for the diffusion case, the corresponding quantities needed are

$$\overline{G}_X(u) = \overline{\Gamma_X * H_X}(u) = \sum_{i=1}^{2} \frac{\frac{q_i}{\alpha_i}}{\frac{q_1}{\alpha_1} + \frac{q_2}{\alpha_2}} \left[\frac{\alpha_i}{\alpha_i - \frac{c_X}{D_X}} e^{-\frac{c_X}{D_X}u} - \frac{\frac{c_X}{D_X}}{\alpha_i - \frac{c_X}{D_X}} e^{-\alpha_i u}\right], \quad u \ge 0,$$

$$\int_0^u \overline{G}_X(t)dt = \sum_{i=1}^{2} \frac{\frac{q_i}{\alpha_i}}{\frac{q_1}{\alpha_1} + \frac{q_2}{\alpha_2}} \left[\frac{\alpha_i}{\alpha_i - \frac{c_X}{D_X}} \frac{1 - e^{-\frac{c_X}{D_X}u}}{\frac{c_X}{D_X}} - \frac{\frac{c_X}{D_X}}{\alpha_i - \frac{c_X}{D_X}} \frac{1 - e^{-\alpha_i u}}{\alpha_i}\right], \quad u \ge 0,$$

and

$$E[L_X^{\star}] = E[N_X]E[L_c^X + L_o^X] = \frac{1}{\theta_X}\left[\frac{\frac{q_1}{\alpha_1^2} + \frac{q_2}{\alpha_2^2}}{\frac{q_1}{\alpha_1} + \frac{q_2}{\alpha_2}} + \frac{D_X}{c_X}\right].$$

The $\beta(u)$,u > 0, for UB_{T_0} and UB_T can be solved by setting Eq. (2.4) equal to 0 with $\beta(0) = 1/e_X(0)$, and UB_{T_0} and UB_T are given from Eqs. (3.11) and (3.12), respectively. By Tsai

(2008), $\psi_X(u) = \theta_X(C_1\ e^{-t_1 u} + C_2\ e^{-t_2 u})$ where t_1 and t_2 satisfy

$$t^2 - \left[(\alpha_1 + \alpha_2) - \frac{1}{1+\theta_X}\frac{1}{\frac{q_1}{\alpha_1} + \frac{q_2}{\alpha_2}}\right]t + \frac{\theta_X}{1+\theta_X}\ \alpha_1\ \alpha_2 = 0,$$

and $C_j = \left\{\frac{t_j}{\frac{q_1}{\alpha_1} + \frac{q_2}{\alpha_2}}\left[\Sigma_{k=1}^{2}\frac{q_k}{(\alpha_k - t_j)^2}\right]\right\}^{-1}$, $j = 1{,}2$. The formula-based expression for $\overline{K}_X(u)$ can be found in Tsai (2003).

When $q_1 = 0.4$, $q_2 = 0.6$, $\alpha_1 = 2$, $\alpha_2 = 0.75$, $u_* = 20$, $\lambda_X = 1$, $\theta_X = 0.2$, *and* $D_X = 0.5$, κ_0 and κ in Eqs. (3.2) and (3.7) can be solved as $\kappa_0 \approx 0.140774$ and $\kappa \approx 0.108002$, respectively. We put curves and numerical values of ψ_X, UB_{T_0}, UB_{L_0}, UB_{C_0}, and UB_{B_0} in Figure 3(a) and Table 1(a), and the ones of $\overline{K}_X$, UB_T, UB_L, UB_C, and UB_B in Figure 3(b) and Table 1(b) for comparison. From Figure 3 we observe that:

- $UB_{B_0}(u)$ and $UB_{C_0}(u)$ do not converge to zero quickly. Though $UB_{L_0}(u)$ is the largest for small u, it is lowered down far more quickly than $UB_{B_0}(u)$ and $UB_{C_0}(u)$ due to an exponential term.
- Among these upper bounds, our $UB_{T_0}(u)$ is the smallest for $u < 13.92$ and $UB_{L_0}(u)$ is slightly lower than $UB_{T_0}(u)$ for $u > 13.92$. Our $UB_{T_0}(u)$ is very close to $\psi_X(u)$ for all $u \le u_* = 20$. It demonstrates that our UB_{T_0} not only can be easily constructed, but also offers a very tight upper bound for ψ_X in this case.
- For the diffusion case, these upper bounds for $\overline{K}_X$ have very similar shapes to the ones for ψ_X except that $UB_C(u)$ increases first for small u then decreases to zero, an unusual situation. The crossing point between $UB_T(u)$ and $UB_L(u)$ occurs at $u \approx 17.74$, larger than 13.92 for the no-diffusion case.

Example 5 *Upper bounds for ruin probabilities from Pareto.*

Let $X \sim$ Pareto(α,τ), $\alpha > 2$, with $f_X(t)$ and $e_X(t)$ given in Example 2, where $e_X(t)$ linearly increases from $\tau/(\alpha-1)$ to infinity. It can be calculated that

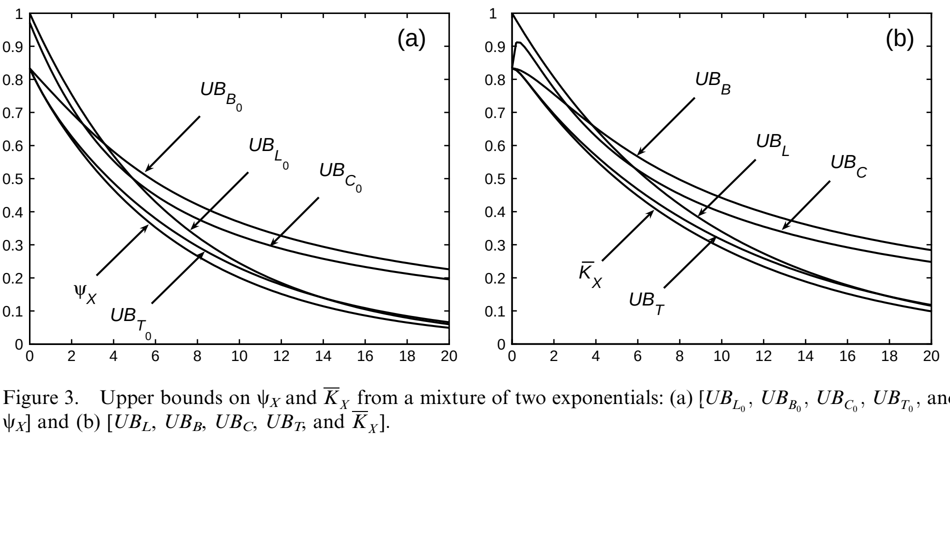

Figure 3. Upper bounds on ψ_X and $\overline{K}_X$ from a mixture of two exponentials: (a) [UB_{L_0}, UB_{B_0}, UB_{C_0}, UB_{T_0}, and ψ_X] and (b) [UB_L, UB_B, UB_C, UB_T, and $\overline{K}_X$].

Table 1. Upper bounds on ψ_X and $\overline{K}_X$ from a mixture of two exponentials.

	(a) ψ_X and its upper bounds					(b) $\overline{K}_X$ and its upper bounds				
u	$\psi_X(u)$	UB_{T_0}	UB_{L_0}	UB_{C_0}	UB_{B_0}	$\overline{K_X}(u)$	UB_T	UB_L	UB_C	UB_B
0	0.8333	0.8333	1.0000	0.9716	0.8333	0.8333	0.8333	1.0000	0.8333	0.8333
2	0.6193	0.6274	0.7546	0.7161	0.6985	0.6895	0.6949	0.8057	0.7726	0.7547
4	0.4670	0.4871	0.5694	0.5538	0.5819	0.5553	0.5691	0.6492	0.6274	0.6531
6	0.3524	0.3793	0.4297	0.4495	0.4904	0.4474	0.4672	0.5231	0.5256	0.5662
8	0.2659	0.2954	0.3243	0.3785	0.4211	0.3605	0.3837	0.4215	0.4526	0.4968
10	0.2007	0.2300	0.2447	0.3273	0.3683	0.2905	0.3151	0.3396	0.3977	0.4417
12	0.1514	0.1792	0.1847	0.2883	0.3271	0.2340	0.2589	0.2736	0.3548	0.3975
14	0.1143	0.1395	0.1393	0.2577	0.2941	0.1886	0.2126	0.2205	0.3203	0.3612
16	0.0862	0.1087	0.1051	0.2330	0.2672	0.1519	0.1747	0.1776	0.2920	0.3310
18	0.0651	0.0846	0.0793	0.2126	0.2448	0.1224	0.1434	0.1431	0.2682	0.3055
20	0.0491	0.0659	0.0599	0.1955	0.2258	0.0986	0.1179	0.1153	0.2480	0.2836

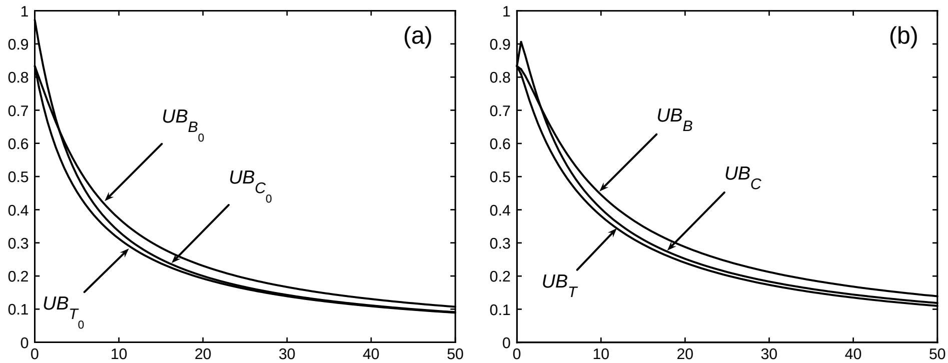

Figure 4. Upper bounds on ψ_X and $\overline{K}_X$ from Pareto: (a) [UB_{B_0}, UB_{C_0}, and UB_{T_0}] and (b) [UB_B, UB_C, and UB_T].

$$\overline{\Gamma}_X(u) = \left[\frac{\tau}{u+\tau}\right]^{\alpha-1}, \qquad \int_0^u \overline{\Gamma}_X(t)dt = \frac{\tau}{\alpha-2}\left\{1-\left[\frac{\tau}{u+\tau}\right]^{\alpha-2}\right\}, \qquad E[L_X^{\bullet}] = \frac{1}{\theta_X}\frac{\tau}{\alpha-2}.$$

For the diffusion case,

$$\int_0^u \overline{G}_X(t)dt = \frac{G_X(u)}{c_X/D_X} + \frac{\tau}{\alpha-2}\left\{1-\left[\frac{\tau}{u+\tau}\right]^{\alpha-2}\right\}, \qquad E[L_X^{\star}] = \frac{1}{\theta_X}\left[\frac{c_X}{D_X}+\frac{\tau}{\alpha-2}\right].$$

There is no explicit expression for $\overline{G}_X(u)$ needed for calculating $UB_C(u)$; in this case, we need some software (for example, MAPLE) to get the numerical value of $\overline{G}_X(u)$ for any given u. The $\beta(u)$ for UB_{T_0} and UB_T is solved as $\beta(0)=1/e_X(0)$ and $\beta(u)=[(\alpha-1)/u]ln(1+u/\tau)$, $u>0$. The expressions for $\psi_X(u)$ and $\overline{K}_X(u)$ are unavailable in the Pareto case; hence we are not able to know how accurate these bounds are. Moreover, κ_0 and κ for respective UB_{L_0} and UB_L do not exist; therefore, $UB_{L_0}(u)$ and $UB_L(u)$ can not be plotted.

Let $\alpha=7$, $\tau=6$, $u_*=50$, $\lambda_X=1$, $\theta_X=0.2$, and $D_X=0.5$. We place curves and numerical values of UB_{T_0}, UB_{C_0}, and UB_{B_0} in Figure 4(a) and Table 2(a), respectively. There is a crossing point at some small u between $UB_{B_0}(u)$ and $UB_{C_0}(u)$, and $UB_{C_0}(u)$ is tighter than $UB_{B_0}(u)$ for u beyond the crossing point. Our $UB_{T_0}(u)$ is the best among these three upper bounds in this case. When there is a diffusion component in the surplus process, the

Table 2. Upper bounds on ψ_X and $\overline{K}_X$ from Pareto.

	(a) upper bonds on ψ_X			(b) upper bonds on $\overline{K}_X$		
u	UB_{T_0}	UB_{C_0}	UB_{B_0}	UB_T	UB_C	UB_B
0	0.8333	0.9722	0.8333	0.8333	0.8333	0.8333
5	0.4545	0.5044	0.5332	0.5326	0.5779	0.6070
10	0.3125	0.3343	0.3733	0.3815	0.4033	0.4454
15	0.2381	0.2502	0.2853	0.2954	0.3101	0.3498
20	0.1923	0.2001	0.2307	0.2401	0.2520	0.2877
25	0.1613	0.1667	0.1935	0.2016	0.2123	0.2443
30	0.1389	0.1429	0.1666	0.1734	0.1834	0.2122
35	0.1220	0.1250	0.1463	0.1519	0.1614	0.1876
40	0.1087	0.1111	0.1304	0.1349	0.1441	0.1681
45	0.0980	0.1000	0.1176	0.1213	0.1302	0.1523
50	0.0893	0.0909	0.1071	0.1100	0.1187	0.1392

orderings among these three upper bounds (see Figure 4(*b)* and Table 2(*b*)) are very similar to the ones for the no-diffusion case. Again, we observe an unusual shape behavior for $\mathrm{UB}_C(u)$.

Note that since both the mean residual lifetime functions of Pareto and a mixture of two exponential distributions are increasing over $[0, \infty)$, both u_**'s* in Examples 4 and 5 can be set equal to ∞. Therefore, our smooth upper bounds $UB_{T_0}(u)$ and $UB_T(u)$ obtained from Eqs. (3.11) and (3.12), respectively, are valid for all $u \geq 0$.

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Appendix A: A technical preliminary

In this appendix, we modify definitions of some orders between two random variables (Shaked & Shanthikumar (2007)). First, let $r_Z(t) = f_Z(t)/\overline{F}_Z(t)$ be the hazard rate (or failure rate) function, $e_Z(t) = E(Z - t|Z > t) = \int_t^\infty \overline{F}_Z(u)du/\overline{F}_Z(t)$ be the mean residual lifetime function, and $\Pi_Z(t) = \int_t^\infty (u - t)dF_Z(u) = \int_t^\infty \overline{F}_Z(u)du$ be the stop-loss transform for a random variable Z.

Definition 1 *The distribution function F_Z is decreasing/increasing failure rate (DFR/IFR) if $r_Z(t)$ is non-increasing/decreasing in t over $\{t:\overline{F}_Z(t)>0\}$ provided that F_Z is absolutely continuous.*

Example 6 *Gamma(α,β) with $\alpha \geq 1$, and Weibull(α,τ) with $\tau \geq 1$ are IFR; Gamma(α,β) with $\alpha \leq 1$, Weibull(α,τ) with $\tau \leq 1$, Pareto(α,β), and a mixture of exponentials are DFR (See Tsai* (2008)).

Definition 2 *The distribution function F_Z is increasing/decreasing mean residual lifetime (IMRL/DMRL) if $e_Z(t)$ is non-decreasing/increasing in t over $\{t:\overline{F}_Z(t)>0\}$ provided that F_Z is absolutely continuous.*

Note that IMRL/DMRL is implied by DFR/IFR.

Definition 3 *X is less than Y in the meaning of the mean residual lifetime order over $[a,b)$, denoted by $X \leq_{mrl[a,b)} Y$, if $e_X(t) \leq e_Y(t)$ for $t \in [a,b)$.*

Definition 4 *X is less than Y in the meaning of the harmonic mean residual lifetime order over $[a,b)$, denoted by $X \leq_{hmrl[a,b)} Y$, if $\{(1/t)\int_0^t [1/e_X(s)]ds\}^{-1} \leq \{(1/t)\int_0^t [1/e_Y(s)]ds\}^{-1}$ for $t \in [a,b)$ and $t > 0$.*

Definition 5 *X is less than Y in the meaning of the stochastic dominance order over $[a,b)$, denoted by $X \leq_{st[a,b)} Y$, if $E[w(X)] \leq (\geq) E[w(Y)]$ for any non-decreasing (non-increasing) real function w on $[a,b)$, or equivalently, $\overline{F}_X(t) \leq \overline{F}_Y(t)$ for $t \in [a,b)$.*

Definition 6 *X is less than Y in the meaning of the stop-loss order over $[a,b)$, denoted by $X \leq_{sl[a,b)} Y$, if $\Pi_X(t) \leq \Pi_Y(t)$ for all $t \in [a,b)$.*

The stop-loss transform can be generalized for a non-negative and decreasing function which is not necessarily a distribution tail as follows. Let $\Omega = \{Q(u),\ u \geq 0 : Q(u) \geq 0$, *decreasing and* $lim_{u\to\infty} Q(u) = 0\}$; the stop-loss transform of $Q \in \Omega$ is defined as $\Pi_Q(t) = -\int_t^\infty (u-t)dQ(u)$. Then, we have the following stop-loss order definition for a pair of functions in Ω (see Cheng & Pai (2003)).

Definition 7 *Suppose Q_1, $Q_2 \in \Omega$, and the stop-loss transforms of Q_1 and Q_2 exist. Q_1 is less than Q_2 in the meaning of the stop-loss order over $[a,b)$, denoted by $Q_1 \leq_{sl[a,b)} Q_2$, if $-\int_t^\infty (u-t)dQ_1(u) = \Pi_{Q_1}(t) \leq \Pi_{Q_2}(t) = -\int_t^\infty (u-t)dQ_2(u)$ for all $t \in [a,b)$.*

Note that when $a = 0$ and $b = \infty$ the ordering definitions above become traditional ones for ordering two random variables. In this case, $[0, \infty)$ following the ordering symbol can be omitted; for example, we denote $X \leq_{st} Y$ instead of $X \leq_{st[0,\infty)} Y$. Moreover, the interval $[a,b)$ *can be changed to* $[a,b]$, (a,b), *or* $(a,b]$ when situations apply, where $0 \leq a < b \leq \infty$.