

On orderings and bounds in a generalized Sparre Andersen risk model

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Abstract

A generalization of the Gerber-Shiu function proposed by Cheung et al. (2009a) is used to derive some ordering properties for certain ruin-related quantities in a Sparre Andersen type risk model. Additional bounds and/or refinements can be obtained by further assuming that the claim size and the interclaim time distributions possess certain reliability properties. Finally, numerical examples are considered to compare the exact solution to the bounds.

Keywords: Gerber-Shiu function, Sparre Andersen risk model, last interclaim time, claim causing ruin, last ladder height, stochastic ordering, NBU, NWU.

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1 Introduction and preliminaries

In this paper, we consider the surplus process of an insurance company $\{U_t, t \geq 0\}$ defined as

$$U_t = u + ct - \sum_{i=1}^{N_t} Y_i, \quad t \geq 0,$$

where $u \geq 0$ is the initial surplus level and $c > 0$ is the incoming premium rate. The claim number process $\{N_t, t \geq 0\}$ is a renewal process defined through the sequence of independent and identically distributed (i.i.d.) positive interclaim times $\{V_i\}_{i=1}^{\infty}$ where V_1 is the time of the first claim and V_i ($i = 2, 3, \dots$) is the time between the $(i - 1)$ -th claim and the i -th claim. Let $K(t) = 1 - \bar{K}(t) = \Pr\{V \leq t\}$ be the cumulative distribution function (c.d.f.) of V , an arbitrary V_i . We further assume

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that $K(t)$ is differentiable for all $t \geq 0$ and hence V has density $k(t) = K'(t)$ and Laplace transform $\tilde{k}(s) = \int_0^\infty e^{-st} k(t) dt$. Also, the claim size random variables (r.v.'s) $\{Y_i\}_{i=1}^\infty$ form a sequence of i.i.d. positive r.v.'s.

In the Sparre Andersen risk model, independence between the sequences $\{V_i\}_{i=1}^\infty$ and $\{Y_i\}_{i=1}^\infty$ is further assumed. Indeed, one can relax this independence assumption by assuming that the pairs $\{(V_i, Y_i); i = 1, 2, \dots\}$ are i.i.d.. Under this more general framework, $\{cV_i - Y_i\}_{i=1}^\infty$ is still an i.i.d. sequence which implies that the surplus process $\{U_t, t \geq 0\}$ retains the Sparre Andersen random walk structure. In what follows, we shall refer to such a generalization of the usual Sparre Andersen model as the *general Sparre Andersen model*.

With (V, Y) being an arbitrary pair of (V_i, Y_i) , it is convenient notationally to specify the joint distribution of (V, Y) by the product of the marginal density of V , namely $k(t)$, and the conditional density of $Y|V$. Thus, for a given $t \geq 0$, we define the c.d.f. $P_t(y) = \Pr\{Y \leq y|V = t\} = 1 - \bar{P}_t(y)$ for $y \geq 0$. Assuming that $P_t(y)$ is differentiable in y ($y \geq 0$) for each fixed $t > 0$, we denote its corresponding density by $p_t(y) = P'_t(y)$. Then, the joint density of (V, Y) can be expressed as $p_t(y)k(t)$. Note that if $P_t(y)$ does not depend on t , the present risk model corresponds to the usual Sparre Andersen model.

Let T be the time of ruin defined as $T = \inf\{t \geq 0 : U_t < 0\}$ with $T = \infty$ if $U_t \geq 0$ for all $t \geq 0$. For convenience, we also introduce the sequence $\{R_n\}_{n=0}^\infty$ where $R_0 = u$ and $R_n = u + \sum_{i=1}^n (cV_i - Y_i)$ ($n = 1, 2, \dots$). From its definition, it is clear that R_n corresponds to the surplus level immediately after the payment of the n -th claim ($n = 1, 2, \dots$). In this paper, our main interest is in the derivation of some ordering properties for certain ruin-related quantities. To this end, we shall make use of the Gerber-Shiu function with a generalized penalty function proposed by Cheung et al. (2009a), namely

$$m_\delta(u) = \mathbb{E} \left[e^{-\delta T} w(U_{T-}, |U_T|, R_{N_T-1}) 1(T < \infty) | U_0 = u \right], \quad u \geq 0, \quad (1)$$

where $\delta \geq 0$, w is the so-called penalty function that depends on the surplus immediately prior to ruin U_{T-} , the deficit at ruin $|U_T|$, and the surplus immediately after the second last claim before ruin R_{N_T-1} , and $1(A)$ is the indicator function of the event A . Throughout the paper, we further assume the positive security loading condition $c\mathbb{E}[V] > \mathbb{E}[Y]$ whenever $\delta = 0$.

The analysis of the famous special case of (1), namely

$$m_{\delta,12}(u) = \mathbb{E} \left[e^{-\delta T} w_{12}(U_{T-}, |U_T|) 1(T < \infty) | U_0 = u \right], \quad u \geq 0,$$

(see Gerber and Shiu (1998)) has been done extensively in recent years in various risk models. The interested reader is referred to Badescu et al. (2009), Boudreault et al. (2006) and Cossette et al. (2008) for some general Sparre Andersen risk models with specific dependence structure between the pair (V, Y) . We remark that the asymptotic ruin probability (for light-tailed claims) in the general Sparre Andersen model has also been studied by Albrecher and Teugels (2006) for an arbitrary dependency structure.

As pointed out by Cheung et al. (2009a), the generalized Gerber-Shiu function (1) naturally leads to the analysis of the final interclaim time before ruin, namely $V_{N_T} = (U_{T-} - R_{N_T-1})/c$, and other ruin-related quantities of interest. A generalized version of (1) (whereby the last minimum surplus level before ruin is incorporated in the penalty function) has also been analyzed by Cheung et al. (2009b) in

the framework of the general Sparre Andersen model. Given that the Gerber-Shiu function (1) plays a key role in our upcoming analysis, an important result regarding its solution form is quoted here (see Cheung (2009) for its derivation in a more general context). The Gerber-Shiu function (1) can be expressed as

$$m_\delta(u) = \beta_\delta(u) + \int_0^\infty \tau_\delta(u, z)\beta_\delta(z) dz, \quad u \geq 0, \quad (2)$$

where

$$\beta_\delta(u) = \int_0^\infty \int_u^\infty w(x, y, u)h_{1,\delta}(x, y|u) dx dy, \quad u \geq 0, \quad (3)$$

is the contribution to $m_\delta(u)$ by ruin occurring on the first claim, and $\tau_\delta(u, z)$ is a non-negative function which may be viewed as a discounted transition density. Here,

$$h_{1,\delta}(x, y|u) = \frac{1}{c}e^{-\frac{\delta(x-u)}{c}}k\left(\frac{x-u}{c}\right)p_{\frac{x-u}{c}}(x+y), \quad x > u; y > 0, \quad (4)$$

is the so-called discounted density of U_{T-} and $|U_T|$ for ruin occurring on the first claim. The representation (2) of $m_\delta(u)$ turns out to be useful for proving the main results of Section 2. For a detailed analysis of the function $\tau_\delta(u, z)$, we refer the interested reader to Cheung (2009), Cheung et al. (2009a) and Willmot and Woo (2009).

When $w \equiv 1$, the Gerber-Shiu function $m_\delta(u)$ reduces to the Laplace transform of the time of ruin $\bar{L}_\delta(u) = E[e^{-\delta T}1(T < \infty)|U_0 = u]$, which is known (see Cheung et al. (2009b)) to satisfy the defective renewal equation

$$\bar{L}_\delta(u) = \phi_\delta \int_0^u \bar{L}_\delta(u-y)f_\delta(y) dy + \phi_\delta \int_u^\infty f_\delta(y) dy, \quad u \geq 0,$$

where $\phi_\delta = \bar{L}_\delta(0) < 1$ and $f_\delta(y)$ ($y > 0$) is the so-called (proper) ladder height density which governs the amount of the first drop of the process $\{U_t, t \geq 0\}$ below its initial level. From Willmot and Lin (2001), for example, $\bar{L}_\delta(u)$ can be represented as the tail of a compound geometric distribution

$$\bar{L}_\delta(u) = \Pr\left\{\sum_{i=1}^{M_\delta} \Theta_{\delta,i} > u\right\}, \quad u \geq 0,$$

where M_δ is a geometric r.v. with probability mass function (p.m.f.) $\Pr\{M_\delta = n\} = (1 - \phi_\delta)\phi_\delta^n$ for $n = 0, 1, \dots$, and $\{\Theta_{\delta,i}\}_{i=1}^\infty$ is a sequence of i.i.d. r.v.'s having the same distribution as a generic r.v. Θ_δ with density $f_\delta(y)$. We remark that the last ladder height at ruin may be expressed when $\delta = 0$ as $X_T + |U_T|$ where $X_T = \inf_{0 \leq s < T} U_s$.

The rest of the paper is structured as follows. In Section 2, the distributions of the last interclaim time before ruin $V_{N_T} = (U_{T-} - R_{N_T-1})/c$, the claim causing ruin $Y_{N_T} = U_{T-} + |U_T|$, and the last ladder height at ruin will be studied in relation to the corresponding marginal distributions of V , Y and Θ_δ respectively. Sufficient conditions for some ordering properties to hold are derived. Section 3 provides additional bounds and/or refinements using reliability properties of V and Y when they are assumed independent. In Section 4, we conclude the paper with some illustrations to evaluate the performance of some bounds derived in Sections 2 and 3.

2 Ordering properties of some ruin-related quantities

In this section, we analyze in more detail the distribution of some quantities of interest in ruin theory. Unless otherwise specified, we assume that $\delta = 0$. Then, Eq. (2) reduces to

$$m_0(u) = \beta_0(u) + \int_0^\infty \tau_0(u, z)\beta_0(z) dz, \quad (5)$$

where, according to (3) and (4),

$$\begin{aligned} \beta_0(u) &= \frac{1}{c} \int_0^\infty \int_u^\infty w(x, y, u) k\left(\frac{x-u}{c}\right) p_{\frac{x-u}{c}}(x+y) dx dy \\ &= \int_0^\infty \int_{u+ct}^\infty w(u+ct, y-u-ct, u) k(t) p_t(y) dy dt. \end{aligned} \quad (6)$$

2.1 The last interclaim time before ruin

In this sub-section, we compare the proper distribution of the last interclaim time before ruin with the distribution of a generic interclaim time V . To this end, we consider the Gerber-Shiu function (1) with $w(x, y, z) = e^{-s(x-z)/c}$. For this choice of penalty function, (6) and (5) respectively become

$$\beta_0(u) = \int_0^\infty e^{-st} k(t) \bar{P}_t(u+ct) dt,$$

and

$$\mathbb{E} [e^{-sV_{N_T}} \mathbf{1}(T < \infty) | U_0 = u] = \int_0^\infty e^{-st} k(t) \left\{ \bar{P}_t(u+ct) + \int_0^\infty \tau_0(u, z) \bar{P}_t(z+ct) dz \right\} dt. \quad (7)$$

Let $g_V(t|u)$ be the (proper) density of $(V_{N_T} | T < \infty)$ for an initial surplus of u . Using (7), one concludes that

$$g_V(t|u) = a_u(t) k(t), \quad t > 0, \quad (8)$$

where

$$a_u(t) = \frac{1}{\psi(u)} \left\{ \bar{P}_t(u+ct) + \int_0^\infty \tau_0(u, z) \bar{P}_t(z+ct) dz \right\}, \quad t > 0, \quad (9)$$

and

$$\psi(u) = \Pr \{T < \infty | U_0 = u\}.$$

From (8), we observe that if $a_u(t)$ is decreasing (i.e. non-increasing) in t for each fixed $u \geq 0$, then $(V_{N_T} | T < \infty)$ is smaller than a generic interclaim time r.v. V in likelihood ratio order, i.e.

$$(V_{N_T} | T < \infty) \leq_{LR} V. \quad (10)$$

Define $\bar{G}_V(t|u) = \int_t^\infty g_V(x|u) dx$. Relation (10) implies that

$$\bar{G}_V(t|u) \leq \bar{K}(t), \quad t > 0, \quad (11)$$

i.e. $V_{N_T}|T < \infty$ is stochastically smaller than V (see e.g. Denuit et al. (2005)).

A closer look at (9) reveals that a sufficient condition for $a_u(t)$ to be decreasing in t for each fixed $u \geq 0$ (leading to the orderings (10) and (11)) is for $\bar{P}_t(z+ct)$ to be decreasing in t for each fixed $z \geq 0$. Such a sufficient condition may not be always easy to check. Fortunately, since $z+ct$ is increasing (i.e. non-decreasing) in t for each fixed $z \geq 0$ and $\bar{P}_t(\cdot)$ is a survival function, a sufficient condition for all the above to hold true is for $\bar{P}_t(y)$ to be decreasing in t for each fixed $y \geq 0$.

Remark 1 *It is instructive to note that the above orderings always hold true in the traditional Sparre Andersen model where $\{V_i\}_{i=1}^\infty$ and $\{Y_i\}_{i=1}^\infty$ are independent (since in such case $\bar{P}_t(\cdot)$ does not depend on t). These generalize the ordering (11) obtained by Cheung et al. (2009a,b) when either the interclaim time or the claim size is assumed to be exponentially distributed.*

Remark 2 *In the literature, the condition that $\bar{P}_t(y)$ is decreasing in t for each fixed $y \geq 0$ is well-documented. Given that $\bar{P}_t(y)$ is the survival function of $(Y|V=t)$, we say that Y is stochastically decreasing in V , denoted by $SD(Y|V)$. The condition $SD(Y|V)$ is indeed a form of negative association for the pair (V, Y) which implies $\text{Cov}(V, Y) \leq 0$ (see e.g. Joag-Dev and Proschan (1983), Lehmann (1966) and Shaked (1977)). In fact, the negative dependence between V and Y under the condition $SD(Y|V)$ explains intuitively why (11) holds true. A negative dependence implies that a short interclaim time is more likely to result in a large claim. As a result, insufficient premium income is available to absorb the upcoming claim which results in ruin. Therefore, ruin is likely to be accompanied by a relatively short interclaim time and this explains (11).*

Example 3 *Consider the dependence structure proposed by Boudreault et al. (2006), i.e.*

$$p_t(y) = e^{-\alpha t} f_1(y) + (1 - e^{-\alpha t}) f_2(y), \quad y > 0,$$

where $\alpha \geq 0$ is a dependence parameter and $f_i(y)$ ($i = 1, 2$) is a proper density function with survival function $\bar{F}_i(y)$ and finite mean μ_i .

If

$$\bar{F}_1(y) \geq \bar{F}_2(y), \quad y \geq 0, \tag{12}$$

it is easy to see that $\bar{P}_t(y)$ is a decreasing function of t for each fixed $y \geq 0$. Thus, (12) is a sufficient condition for the orderings (10) and (11) to hold.

For the same risk model but with the inequality in (12) reversed, then $\bar{P}_t(y)$ is an increasing function of t for each fixed $y \geq 0$. The sufficient condition is no longer met. Hence, further examination is needed to determine whether the orderings (10) and (11) remain valid.

2.2 The claim causing ruin

We now consider a penalty function of the form $w(x, y, z) = e^{-s(x+y)}$ which leads to the Laplace transform of the claim causing ruin Y_{N_T} . Hence, (6) and (5) can respectively be reduced to

$$\beta_0(u) = \int_u^\infty e^{-sy} \left\{ \int_0^{\frac{y-u}{c}} p_t(y)k(t) dt \right\} dy,$$

and

$$\begin{aligned} \mathbb{E} [e^{-sY_{N_T}} \mathbf{1}(T < \infty) | U_0 = u] &= \int_u^\infty e^{-sy} \left\{ \int_0^{\frac{y-u}{c}} p_t(y)k(t) dt \right\} dy \\ &+ \int_0^\infty \left\{ \int_z^\infty e^{-sy} \int_0^{\frac{y-z}{c}} p_t(y)k(t) dt dy \right\} \tau_0(u, z) dz. \end{aligned} \quad (13)$$

With a slight abuse of notation, an application of Bayes theorem yields

$$p_t(y)k(t) = p(y)k_y(t), \quad (14)$$

where $p(\cdot)$ is the marginal density of the generic claim size Y and $k_y(\cdot)$ is the density of $(V | Y = y)$. Substituting (14) into (13) followed by some simple manipulations leads to

$$\begin{aligned} &\mathbb{E} [e^{-sY_{N_T}} \mathbf{1}(T < \infty) | U_0 = u] \\ &= \int_u^\infty e^{-sy} p(y) K_y \left(\frac{y-u}{c} \right) dy + \int_0^\infty \left\{ \int_u^\infty e^{-sy} p(y) K_y \left(\frac{y-z}{c} \right) dy \right\} \tau_0(u, z) dz \\ &= \int_0^\infty e^{-sy} p(y) \left\{ K_y \left(\frac{y-u}{c} \right) dy + \int_0^z K_y \left(\frac{y-z}{c} \right) \tau_0(u, z) dz \right\} dy, \end{aligned} \quad (15)$$

where $K_y(t) = \int_0^t k_y(x) dx$. Let $g_Y(y|u)$ be the (proper) density of $(Y_{N_T} | T < \infty)$ for an initial surplus of u . From (15), it follows that

$$g_Y(y|u) = b_u(y)p(y), \quad y > 0, \quad (16)$$

where

$$b_u(y) = \frac{1}{\psi(u)} \left\{ K_y \left(\frac{y-u}{c} \right) + \int_0^z K_y \left(\frac{y-z}{c} \right) \tau_0(u, z) dz \right\}, \quad y > 0. \quad (17)$$

Similar as in Section 2.1, one observes from (16) and (17) that if $b_u(y)$ is increasing in y for each fixed $u \geq 0$, then $(Y_{N_T} | T < \infty)$ is larger than a generic claim size r.v. Y in likelihood ratio order, i.e.

$$(Y_{N_T} | T < \infty) \geq_{LR} Y, \quad (18)$$

which implies the stochastic ordering

$$\bar{G}_Y(y|u) \geq \bar{P}(y), \quad y > 0, \quad (19)$$

where $\bar{G}_Y(y|u) = \int_y^\infty g_Y(x|u) dx$ and $\bar{P}(y) = \int_y^\infty p(x) dx$. Examination of (17) reveals that a sufficient condition for $b_u(y)$ to be increasing in y for each fixed $u \geq 0$ is for $K_y((y-z)/c)$ to be increasing in y for each fixed $z \geq 0$. This in turn has sufficient condition that $K_y(t)$ is increasing in y for each fixed $t \geq 0$. In the literature, the latter condition is known as $SD(V|Y)$, i.e. V is stochastically decreasing in Y . The same probabilistic interpretation as for the ordering (11) given in Remark 2 applies. It is also clear that the orderings (18) and (19) always hold true when $\{V_i\}_{i=1}^\infty$ and $\{Y_i\}_{i=1}^\infty$ are independent.

Example 4 (Reverse dependence structure to Example 3) Let

$$k_y(t) = e^{-\kappa y} g_1(t) + (1 - e^{-\kappa y}) g_2(t), \quad t > 0,$$

where $\kappa \geq 0$ and $g_i(t)$ ($i = 1, 2$) is a (proper) density function with survival function $\bar{G}_i(t)$. In this dependence risk model, a sufficient condition for the orderings (18) and (19) to hold is

$$\bar{G}_1(t) \geq \bar{G}_2(t), \quad t \geq 0.$$

Remark 5 In fact, the same condition $SD(Y|V)$ ($SD(V|Y)$) is sufficient to guarantee the orderings (10) and (11) ((18) and (19)) to hold in the general Sparre Andersen risk model even when the premium rate is allowed to depend on the level of the surplus. The derivation essentially follows along the same lines and is omitted here. For a detailed description of the model with surplus-dependent premium rate, see Cheung (2009).

2.3 The last ladder height

In this sub-section, we compare for an arbitrary value of $\delta \geq 0$ the (normalized) discounted distribution of the last ladder height at ruin to the distribution of Θ_δ . To begin, by recalling that $X_T = \inf_{0 \leq s < T} U_s$ represents the minimum surplus level before ruin, we define the discounted density of the last ladder height at ruin $\nu_\delta(y|u)$ via its Laplace transform, namely

$$\mathbb{E} \left[e^{-\delta T - z(X_T + |U_T|)} \mathbf{1}(T < \infty) | U_0 = u \right] = \int_0^\infty e^{-zy} \nu_\delta(y|u) dy, \quad u \geq 0. \quad (20)$$

Given that (20) at $z = 0$ yields $\int_0^\infty \nu_\delta(y|u) dy = \bar{L}_\delta(u)$, the normalized density corresponding to $\nu_\delta(y|u)$ is given (e.g. Cheung et al. (2009b)) by

$$\frac{\nu_\delta(y|u)}{\bar{L}_\delta(u)} = r_u(y) f_\delta(y), \quad y > 0, \quad (21)$$

where

$$r_u(y) = \begin{cases} \frac{\phi_\delta}{1-\phi_\delta} \frac{\bar{L}_\delta(u-y) - \bar{L}_\delta(u)}{\bar{L}_\delta(u)}, & y < u \\ \frac{\phi_\delta}{1-\phi_\delta} \frac{1 - \bar{L}_\delta(u)}{\bar{L}_\delta(u)}, & y \geq u \end{cases}.$$

Clearly, $r_u(y)$ is increasing in y for each fixed $u \geq 0$ since $\bar{L}_\delta(\cdot)$ is a compound geometric tail. Therefore, if Γ_δ denotes a generic r.v. with (proper) density (21), then

$$\Gamma_\delta \geq_{LR} \Theta_\delta, \quad (22)$$

which implies the stochastic ordering

$$\bar{G}_{\Gamma_\delta}(y|u) \geq \bar{F}_\delta(y), \quad y > 0, \quad (23)$$

where $\bar{G}_{\Gamma_\delta}(y|u) = \int_y^\infty (\nu_\delta(x|u) / \bar{L}_\delta(u)) dx$ and $\bar{F}_\delta(y) = \int_y^\infty f_\delta(x) dx$. We remark that the ordering (23) has been proved by Cheung et al. (2009b) when $\delta = 0$.

Remark 6 Unlike the orderings for the last interclaim time before ruin and the claim causing ruin, the orderings (22) and (23) are valid regardless of the form of the joint distribution of (V, Y) . Indeed, as one would have expected, the ladder heights depend on the distribution of $cV - Y$ which represents the increment of the process between claims. Hence, the form of the dependence between V and Y is irrelevant here. For the case $\delta = 0$, Γ_0 has the same distribution as $(X_T + |U_T| | T < \infty)$ which is the last ladder height at ruin given that ruin occurs. The ordering (23) is intuitive given that Γ_0 is the ladder height which causes ruin and should be (on average) larger than the generic ladder height Θ_0 .

3 Some bounds obtainable by reliability properties

Additional bounds and/or refinements are obtainable using reliability properties of $K(t)$ and $P(y)$ when the generic r.v.'s V and Y are independent. For instance, the following theorem generalizes results in Cheung et al. (2009a,b).

Theorem 7 If $K(t)$ is new worse (better) than used or NWU (NBU) (i.e. $\bar{K}(t_1+t_2) \geq (\leq) \bar{K}(t_1)\bar{K}(t_2)$ for all $t_1, t_2 \geq 0$), and there exists a function $\bar{F}(y)$ on $[0, \infty)$ such that

$$\bar{P}(y_1 + y_2) \leq (\geq) \bar{P}(y_1)\bar{F}(y_2), \quad y_1, y_2 \geq 0, \quad (24)$$

the survival function of $(V_{N_T} | T < \infty)$ satisfies

$$\bar{G}_V(t|u) \leq (\geq) \bar{F}(ct)\bar{K}(t), \quad t \geq 0. \quad (25)$$

Proof. From (8),

$$\bar{G}_V(t|u) = \int_t^\infty a_u(x)k(x)dx.$$

Using integration by parts, one obtains

$$\bar{G}_V(t|u) = a_u(t)\bar{K}(t) + \int_0^\infty a'_u(x+t)\bar{K}(x+t)dx.$$

Using the fact that $a'_u(t) \leq 0$ together with the NWU (NBU) property of $K(t)$ leads to

$$\bar{G}_V(t|u) \leq (\geq) \bar{K}(t) \left\{ a_u(t) + \int_0^\infty a'_u(x+t)\bar{K}(x) dx \right\}.$$

Integrating by parts the right-hand side of the above equation yields

$$\bar{G}_V(t|u) \leq (\geq) \bar{K}(t) \int_0^\infty a_u(x+t)k(x) dx. \quad (26)$$

Also, from the definition (9) of $a_u(t)$, it is clear that, under (24),

$$a_u(x+t) \leq (\geq) \bar{F}(ct)a_u(x), \quad x, t > 0. \quad (27)$$

The substitution of (27) into (26) together with the fact that $\bar{G}_V(0|u) = \int_0^\infty a_u(x)k(x)dx = 1$ yields (25). ■

Note that when $K(t)$ is NWU, if $\bar{P}(y)$ and $\bar{F}(y)$ satisfy (24), Eq. (25) results in an improved bound over (11). In contrast, when $K(t)$ is NBU, if $\bar{P}(y)$ and $\bar{F}(y)$ satisfy (24), then (25) and (11) lead to a two-sided bound for $\bar{G}_V(t|u)$, namely

$$\bar{F}(ct)\bar{K}(t) \leq \bar{G}_V(t|u) \leq \bar{K}(t), \quad t \geq 0.$$

Thus, given the reliability property of $K(t)$, it remains to choose $\bar{F}(y)$ such that (24) holds true. The following three choices are suggested depending on the properties of $P(y)$.

1. If $P(y)$ is NBU (NWU), then a choice of $\bar{F}(y)$ would be $\bar{P}(y)$.
2. If $P(y)$ is used worse (better) than aged or UWA (UBA) (which includes the increasing (decreasing) mean residual lifetime or IMRL (DMRL) class), then a choice of $\bar{F}(y)$ would be $e^{-y/r_P(\infty)}$, where $r_P(\infty) = \lim_{y \rightarrow \infty} E[Y - y | Y > y]$ is the mean residual lifetime of Y . Recall that the c.d.f. $P(y)$ is said to be UWA (UBA) if $0 < r_P(\infty) < \infty$, and $\bar{P}(y_1 + y_2) \leq (\geq) \bar{P}(y_1)e^{-y_2/r_P(\infty)}$ for all $y_1, y_2 \geq 0$. See e.g. Willmot and Cai (2000).
3. If $\mu_P(y) \geq (\leq) \mu$ for $y \geq 0$, where $\mu_P(y)$ is the failure rate function corresponding to the c.d.f. $P(y)$, then a choice of $\bar{F}(y)$ would be $e^{-\mu y}$.

It is interesting to note that it is possible to apply Theorem 7 regardless of whether $K(t)$ and $P(y)$ has the same or opposite reliability properties. The reliability property of decreasing (increasing) failure rate or DFR (IFR) implies NWU (NBU) and UWA (UBA). Therefore, when $K(t)$ is DFR (IFR) while $P(y)$ is IFR (DFR), one could choose $\bar{F}(y)$ to be $\bar{P}(y)$ to obtain an improved bound (a two-sided bound). On the other hand, when both $K(t)$ and $P(y)$ are DFR (IFR), then $\bar{F}(y)$ can be chosen to be $e^{-y/r_P(\infty)}$ to get an improved bound (a two-sided bound). Section 4 demonstrates these ideas with four examples.

4 Numerical illustrations

In this section, four numerical examples are considered to evaluate the performance of the bounds proposed in Sections 2 and 3 for the survival function $\bar{G}_V(t|u)$ of the last interclaim time $V_{N_T} = (U_{T^-} - R_{N_T-1})/c$. These bounds are compared to the exact value of $\bar{G}_V(t|u)$, which is computed as the integrated tail of (8). From (9), we remark that $\bar{G}_V(t|u)$ depends on both $\tau_0(u, z)$ and $\psi(u)$. With the specific interclaim time distributions assumed in the upcoming examples, it is known from Willmot and Woo (2009) that $\tau_0(u, z)$ can be evaluated from $\psi'(u)$. The ruin probability $\psi(u)$ can in turn be computed using the algorithms of Landriault and Willmot (2008) for the specific claim size distributions considered below. In all these examples, the r.v.'s $\{V_i\}_{i=1}^{\infty}$ and $\{Y_i\}_{i=1}^{\infty}$ are assumed to be independent, and the generic r.v.'s V and Y both have a mean of 1. In addition, a premium rate of $c = 1.5$ and an initial surplus level of $u = 5$ are assumed.

Example 8 ($K(t)$ is DFR and $P(y)$ is IFR) *The interclaim time is assumed to be a mixture of two exponentials with density*

$$k(t) = \frac{1}{3} \left(\frac{1}{2} e^{-\frac{1}{2}t} \right) + \frac{2}{3} (2e^{-2t}), \quad t > 0. \quad (28)$$

Also, the claim size is assumed to be a sum of two exponentials (of means $4/5$ and $1/5$) with density

$$p(y) = \frac{4}{3} \left(\frac{5}{4} e^{-\frac{5}{4}y} \right) - \frac{1}{3} (5e^{-5y}), \quad y > 0. \quad (29)$$

It can be shown that $0 \leq \mu_P(y) \leq 5/4$ for $y \geq 0$. An application of Theorem 7 with $\bar{F}(y)$ chosen according to the 1st or the 3rd item of Section 3 results in upper bounds for $\bar{G}_V(t|u)$. First, note that the upper bound resulting from the application of the 3rd item is identical to the ordering bound (11) given that μ shall be chosen to be 0. Figure 1 compares the exact value of $\bar{G}_V(t|5)$ to two of its upper bounds: ‘Ordering bound’ refers to the bound obtained via (11), and ‘Improved bound 1’ is obtained by setting $\bar{F}(y) = \bar{P}(y)$.

INSERT FIGURE 1

For this example, one observes that ‘Improved bound 1’ performs significantly better than the ‘Ordering bound’.

Example 9 ($K(t)$ is IFR and $P(y)$ is DFR) The interclaim time is assumed to be a sum of two exponentials (of means $2/3$ and $1/3$) with density

$$k(t) = 2 \left(\frac{3}{2} e^{-\frac{3}{2}t} \right) - 3e^{-3t}, \quad t > 0. \quad (30)$$

Also, the claim size is assumed to be a mixture of two exponentials with density

$$p(y) = \frac{3}{5} \left(\frac{3}{4} e^{-\frac{3}{4}y} \right) + \frac{2}{5} (2e^{-2y}), \quad y > 0. \quad (31)$$

It can easily be verified that $3/4 \leq \mu_P(y) \leq 5/4$ for $y \geq 0$. Hence, Theorem 7 can be applied to obtain lower bounds for $\bar{G}_V(t|5)$. Here, we choose $\bar{F}(y)$ via the 1st or the 3rd item (with $\mu = 5/4$) of Section 3. Figure 2 depicts the exact value of $\bar{G}_V(t|5)$ together with three of its bounds, namely the ‘Ordering bound’ resulting from (11), as well as ‘Lower bound 1’ and ‘Lower bound 3’ obtained from Theorem 7 by replacing $\bar{F}(y)$ by $\bar{P}(y)$ and $e^{-(5/4)y}$ respectively.

INSERT FIGURE 2

For this example, we observe that ‘Lower bound 1’ provides a better lower bound than ‘Lower bound 3’.

Example 10 ($K(t)$ is DFR and $P(y)$ is DFR) The interclaim time and the claim size are assumed to have densities (28) and (31) respectively. It can be verified that $r_P(\infty) = 4/3$. Theorem 7 can be applied to obtain tighter upper bounds than (11) for $\bar{G}_V(t|5)$ by choosing $\bar{F}(y)$ via the 2nd or the 3rd item of Section 3. In applying the 3rd item, the best upper bound is obtained by choosing $\mu = 3/4$. The

resulting bound is identical to the one obtained via the application of item 2. Thus, Figure 3 compares the exact value of $\overline{G}_V(t|5)$ to the ‘Ordering bound’ (11) and the ‘Improved bound 2 or 3’ obtained by setting $\overline{F}(y) = e^{-(3/4)y}$ in Theorem 7.

INSERT FIGURE 3

In this example, ‘Improved bound 2 or 3’ performs significantly better than the ‘Ordering bound’.

Example 11 ($K(t)$ is IFR and $P(y)$ is IFR) The interclaim time and the claim size are assumed to have densities (30) and (29) respectively. We find that $r_P(\infty) = 4/5$. An application of Theorem 7 results in a two-sided bound for $\overline{G}_V(t|5)$. Here we choose $\overline{F}(y)$ according to the 2nd or the 3rd item of Section 3. For the 3rd item, the best lower bound is obtained when $\mu = 5/4$. The resulting bound is identical to the one obtained via the application of item 2. Figure 4 shows the exact value of $\overline{G}_V(t|5)$ together with its upper bound ‘Ordering bound’ obtained from (11), and its ‘Lower bound 2 or 3’ obtained from Theorem 7 by setting $\overline{F}(y) = e^{-(5/4)y}$.

INSERT FIGURE 4

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