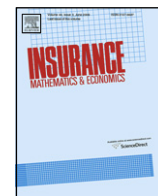




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# Structural properties of Gerber–Shiu functions in dependent Sparre Andersen models

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## ABSTRACT

The structure of various Gerber–Shiu functions in Sparre Andersen models allowing for possible dependence between claim sizes and interclaim times is examined. The penalty function is assumed to depend on some or all of the surplus immediately prior to ruin, the deficit at ruin, the minimum surplus before ruin, and the surplus immediately after the second last claim before ruin. Defective joint and marginal distributions involving these quantities are derived. Many of the properties in the Sparre Andersen model without dependence are seen to hold in the present model as well. A discussion of Lundberg's fundamental equation and the generalized adjustment coefficient is given, and the connection to a defective renewal equation is considered. The usual Sparre Andersen model without dependence is also discussed, and in particular the case with exponential claim sizes is considered.

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## 1. Introduction and background

We consider the surplus process  $\{U_t, t \geq 0\}$  defined by  $U_t = u + ct - \sum_{i=1}^{N_t} Y_i$ , where  $u (u \geq 0)$  is the initial surplus and  $\sum_{i=1}^0 = 0$ . The claim number process  $\{N_t, t \geq 0\}$  is a renewal process defined via the sequence of independent and identically distributed (i.i.d.) interclaim times  $\{V_i\}_{i=1}^{\infty}$  with  $V_1$  the time of the first claim and  $V_i$  for  $i = 2, 3, \dots$  the time between the  $(i-1)$ th claim and the  $i$ th claim. Let  $K(t) = 1 - \bar{K}(t) = \Pr(V \leq t)$  where  $V$  is an arbitrary  $V_i$ , and we assume that  $K(t)$  is differentiable with density  $k(t) = K'(t)$ . Also, the claim size random variables (r.v.'s)  $\{Y_i\}_{i=1}^{\infty}$ , with  $Y_i$  the size of the  $i$ th claim, are assumed to form a sequence of i.i.d. r.v.'s.

In this paper, we assume that the pairs  $\{(V_i, Y_i); i = 1, 2, \dots\}$  are i.i.d., so that  $\{cV_i - Y_i; i = 1, 2, \dots\}$  is also an i.i.d. sequence which implies that the surplus process  $\{U_t, t \geq 0\}$  retains the Sparre Andersen random walk structure. It is convenient notationally to specify the joint distribution of  $(V_i, Y_i)$  by the product of the marginal density  $k(t)$  and the conditional density of  $Y_i$  given  $V_i$ . With  $(V, Y)$  being an arbitrary  $(V_i, Y_i)$ , we let  $P_t(y) =$

$\Pr(Y \leq y | V = t) = 1 - \bar{P}_t(y)$  for  $y > 0$ . Let  $p_t(y) = P'_t(y)$  be the conditional density, so that the joint density of  $(V, Y)$  is given by  $p_t(y)k(t)$ . In what follows, it is also convenient to introduce the conditional Laplace transform

$$\tilde{p}_t(s) = \int_0^{\infty} e^{-sy} p_t(y) dy.$$

It is instructive to note that the assumptions of absolute continuity are not necessary and are simply made for ease of exposition. To complete the definition of  $\{U_t, t \geq 0\}$ , we define  $c$  ( $c > 0$ ) to be the premium rate per unit time which is assumed to satisfy the positive security loading condition, namely  $cE[V] > E[Y]$ .

The classical Gerber–Shiu discounted penalty function is defined by

$$m_{\delta, 12}(u) = E[e^{-\delta T} w_{12}(U_{T-}, |U_T|) 1(T < \infty) | U_0 = u], \quad (1)$$

where  $T = \inf\{t \geq 0 : U_t < 0\}$  with  $T = \infty$  if  $U_t \geq 0$  for all  $t \geq 0$ , i.e.  $T$  is the time of ruin. Also,  $U_{T-}$  is the surplus immediately prior to ruin,  $|U_T|$  is the deficit at ruin,  $w_{12}(x, y)$  satisfies mild integrable conditions,  $1(A)$  is the usual indicator function of the event  $A$ , and  $\delta$  (often interpreted as a force of interest) is assumed to be nonnegative.

The Gerber–Shiu function (1) has been studied extensively in recent years for models of this nature. The usual Sparre Andersen model assumes independence between  $V$  and  $Y$ , and may be

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recovered with  $P_t(y) = P(y)$  for all  $t > 0$ . Boudreault et al. (2006) considered the dependent Poisson risk model with  $K(t) = 1 - e^{-\lambda t}$  and  $P_t(y) = e^{-\beta t} p^{(1)}(y) + (1 - e^{-\beta t}) p^{(2)}(y)$  where  $p^{(1)}(y)$  and  $p^{(2)}(y)$  are “usual” and “severe” claim size distribution functions, respectively. Cossette et al. (2008) also used  $K(t) = 1 - e^{-\lambda t}$ , but with  $P_t(y) = C(P(y), 1 - e^{-\lambda t}) / (1 - e^{-\lambda t})$ , where  $C(u, v)$  is a generalized Farlie–Gumbel–Morgenstern copula. Badescu et al. (2009) assumed a bivariate phase-type distribution for  $(V, Y)$ , and Albrecher and Teugels (2006) examined asymptotics for ruin probabilities for the present model. A similar dependency structure is also examined by Albrecher and Boxma (2004). These models retain many of the properties of the independent case insofar as the Gerber–Shiu analysis is concerned, and the aim of this paper is to examine these phenomena in more detail within the context of the more general dependency model described above. Insight into the nature of the special cases (both independent and dependent) follows as a by-product.

In this paper we generalize (1) as follows. First define  $X_t = \inf_{0 \leq s < t} U_s$  to be the minimum surplus before time  $t$ . Therefore,  $X_T$  is the minimum surplus before ruin. Second, let  $R_n = u + \sum_{i=1}^n (cV_i - Y_i)$  for  $n = 1, 2, \dots$ , and define  $R_0 = u$ . Clearly,  $R_n$  is the surplus immediately following the  $n$ th claim if  $n \geq 1$ , and  $R_{N_T-1}$  is the surplus immediately after the second last claim before ruin occurs if  $N_T > 1$ , and  $R_{N_T-1} = u$  if ruin occurs on the first claim (i.e.  $N_T = 1$ ). Note that  $R_{N_T-1}$  may or may not equal  $X_T$ . Then we generalize (1) to

$$m_\delta(u) = E \left[ e^{-\delta T} w(U_{T-}, |U_T|, X_T, R_{N_T-1}) \mathbf{1}(T < \infty) | U_0 = u \right]. \quad (2)$$

As in the case of the traditional Gerber–Shiu function (1), for the remainder of the paper we shall assume that the penalty function  $w(x, y, z, v)$  satisfies some mild integrable conditions, i.e.  $w(x, y, z, v)$  is such that the expectation in (2) is finite. We remark that the introduction of  $X_T$  allows for the analysis of the last ladder height before ruin, namely  $X_T + |U_T|$ . The analysis involving  $X_T$  has been considered in a Levy process setting by Doney and Kyprianou (2006). Also, the last interclaim time before ruin is  $V_{N_T} = (U_{T-} - R_{N_T-1})/c$ , a quantity which has been studied by Cheung et al. (in press) in the classical compound Poisson risk model (with  $K(t) = 1 - e^{-\lambda t}$ ) via the Gerber–Shiu function

$$m_{\delta,124}(u) = E \left[ e^{-\delta T} w_{124}(U_{T-}, |U_T|, R_{N_T-1}) \mathbf{1}(T < \infty) | U_0 = u \right], \quad (3)$$

a special case of (2) with  $w(x, y, z, v) = w_{124}(x, y, v)$ . Thus (2) allows for the analysis of the last pair  $(V_{N_T}, Y_{N_T})$  before ruin, and we remark that the claim causing ruin  $Y_{N_T} = U_{T-} + |U_T|$  has been studied on numerous occasions, beginning with Dufresne and Gerber (1988).

In the next section we examine the mathematical structure of the Gerber–Shiu function (2) as well as the particular special cases

$$m_{\delta,123}(u) = E \left[ e^{-\delta T} w_{123}(U_{T-}, |U_T|, X_T) \mathbf{1}(T < \infty) | U_0 = u \right], \quad (4)$$

$$m_{\delta,23}(u) = E \left[ e^{-\delta T} w_{23}(|U_T|, X_T) \mathbf{1}(T < \infty) | U_0 = u \right], \quad (5)$$

$$m_{\delta,2}(u) = E \left[ e^{-\delta T} w_2(|U_T|) \mathbf{1}(T < \infty) | U_0 = u \right], \quad (6)$$

and

$$\bar{G}_\delta(u) = E \left[ e^{-\delta T} \mathbf{1}(T < \infty) | U_0 = u \right], \quad (7)$$

which correspond to the successively simplified penalty functions given by  $w(x, y, z, v) = w_{123}(x, y, z)$ ,  $w(x, y, z, v) = w_{23}(y, z)$ ,  $w(x, y, z, v) = w_2(y)$ , and  $w(x, y, z, v) = 1$ , respectively. In particular, we demonstrate that all the Gerber–Shiu functions considered satisfy defective renewal equations, each of which

has associated compound geometric tail (in the sense of Willmot and Lin (2001, Section 9.1)) given by (7). The same basic relationships hold for the present model involving dependence as for the more commonly used case with  $V_i$  independent of  $Y_i$ . In Section 3, the results of Section 2 are used to derive various joint and marginal distributions, and in particular the distribution of the last ladder height before ruin is derived and is shown to be stochastically larger than other ladder heights. In Section 4, some observations concerning Lundberg’s fundamental equation and the generalized adjustment coefficient are made, and the direct role of Lundberg’s equation in the derivation of the defective renewal equation is examined. These observations are of interest in the independent case as well. Finally, in Section 5, some further remarks concerning the independent case are made, and the case with exponential claims is considered in some detail. In particular, the joint Laplace transform of  $(T, U_{T-}, |U_T|, X_T, R_{N_T-1})$  is derived with exponential claim sizes, and the last interclaim time  $V_{N_T}$  before ruin is shown to have an Esscher transformed distribution which is stochastically dominated by a generic interclaim time distribution.

**2. Defective renewal equation and compound geometric properties**

To begin the analysis, we first examine the nature of the joint distribution of the time of ruin  $T$ , the surplus prior to ruin  $U_{T-}$ , the deficit at ruin  $|U_T|$ , and  $R_{N_T-1}$ . If ruin occurs on the first claim, then the surplus ( $x$ ) and the time ( $t$ ) are related by  $x = u + ct$ , or equivalently  $t = (x - u) / c$ . Once the surplus  $x$  has been reached, a claim of size  $x + y$  results in a deficit of  $y$ . The density is thus  $k(t)p_t(x + y)$  where  $t = (x - u) / c$ . Therefore, a change of variables from  $t$  to  $x$  implies that the joint defective density of the surplus prior to ruin ( $x$ ) and the deficit at ruin ( $y$ ) for ruin occurring on the first claim is given by

$$h_1^*(x, y | u) = \frac{1}{c} k \left( \frac{x - u}{c} \right) p_{\frac{x-u}{c}}(x + y), \quad (8)$$

and in this case the time of ruin  $T$  is  $(x - u) / c$  and  $R_{N_T-1}$  equals  $u$ .

If ruin occurs on claims subsequent to the first,  $T$  and  $R_{N_T-1}$  are no longer simple functions of  $U_{T-}$  and  $|U_T|$ , and we denote the joint defective density of the time of ruin ( $t$ ), the surplus before ruin ( $x$ ), the deficit at ruin ( $y$ ), and the surplus after the second last claim ( $v$ ), by  $h_2^*(t, x, y, v | u)$  for  $v < x$ . See Cheung et al. (in press) for further discussion of this density in the classical compound Poisson risk model.

We now employ the argument of Gerber and Shiu (1998) to obtain an integral equation for  $m_\delta(u)$ . We will thus condition on the first drop in the surplus to a value below its initial level of  $u$ . The density of this first drop for a drop on the first claim is  $h_1^*(x, y | 0)$ , where  $x$  refers to the surplus level above  $u$  just before the drop (i.e. the surplus reaches  $x + u$ ), and  $y$  is the drop below  $u$ , so that the surplus level after the drop is  $u - y$ . The time of this drop is  $x/c$ . If  $y > u$ , then ruin occurs on the first drop, and in this case  $U_{T-} = x + u$ ,  $|U_T| = y - u$ ,  $X_T = u$ , and  $R_{N_T-1} = u$ . If  $y < u$  then ruin does not occur, and the process begins anew (probabilistically) beginning at the surplus level  $u - y$ . If the drop in surplus below  $u$  does not occur on the first claim, then the density is  $h_2^*(t, x, y, v | 0)$ . Again, ruin occurs if  $y > u$ , and in this case  $U_{T-} = x + u$ ,  $|U_T| = y - u$ ,  $X_T = u$ , and  $R_{N_T-1} = v + u$ . Similarly, if  $y < u$  then ruin does not occur, and the process continues from the new surplus level of  $u - y$ . Summing (integrating) over all values of  $t, x, y$ , and  $v$  results in the integral equation satisfied by (2), namely

$$m_\delta(u) = \int_0^u m_\delta(u - y) \left\{ \int_0^\infty h_{1,\delta}^*(x, y | 0) dx + \int_0^\infty \int_0^x h_{2,\delta}^*(x, y, v | 0) dv dx \right\} dy + v_\delta(u), \quad (9)$$

where

$$h_{1,\delta}^*(x, y | u) = e^{-\frac{\delta(x-u)}{c}} h_1^*(x, y | u) \tag{10}$$

and

$$h_{2,\delta}^*(x, y, v | u) = \int_0^\infty e^{-\delta t} h_2^*(t, x, y, v | u) dt \tag{11}$$

are “discounted” densities. In this case,  $v_\delta(u)$  is the contribution due to ruin on the first drop and is given by

$$v_\delta(u) = \int_u^\infty \int_0^\infty w(x+u, y-u, u, u) h_{1,\delta}^*(x, y | 0) dx dy + \int_u^\infty \int_0^\infty \int_0^x w(x+u, y-u, u, v+u) \times h_{2,\delta}^*(x, y, v | 0) dv dx dy. \tag{12}$$

Using (8), (12) may also be written as

$$v_\delta(u) = \int_0^\infty e^{-\delta t} \left\{ \int_0^\infty w(u+ct, y, u, u) \times p_t(y+ct+u) dy \right\} k(t) dt + \int_u^\infty \int_0^\infty \int_0^x w(x+u, y-u, u, v+u) \times h_{2,\delta}^*(x, y, v | 0) dv dx dy. \tag{13}$$

We now examine the structure of (9) in more detail.

First, note that the discounted (marginal if  $\delta = 0$ ) density of  $U_T$  and  $|U_T|$  is obtained by summing and integrating over all values of  $t$  and  $v$ , yielding

$$h_\delta(x, y | u) = h_{1,\delta}^*(x, y | u) + \int_0^x h_{2,\delta}^*(x, y, v | u) dv. \tag{14}$$

Using (14) with  $u = 0$ , (9) may be re-expressed as

$$m_\delta(u) = \int_0^u m_\delta(u-y) \left\{ \int_0^\infty h_\delta(x, y | 0) dx \right\} dy + v_\delta(u). \tag{15}$$

It is not hard to see from (14) that (1) may be written as

$$m_{\delta,12}(u) = \int_0^\infty \int_0^\infty w_{12}(x, y) h_\delta(x, y | u) dx dy. \tag{16}$$

Thus, letting

$$\phi_\delta = \int_0^\infty \int_0^\infty h_\delta(x, y | 0) dx dy, \tag{17}$$

it is clear from (16) with  $w_{12}(x, y) = 1$  and (1) that  $\phi_\delta = E[e^{-\delta T} 1(T < \infty) | U_0 = 0] < 1$ . Also, define the ladder height density

$$f_\delta(y) = \frac{1}{\phi_\delta} \int_0^\infty h_\delta(x, y | 0) dx, \tag{18}$$

which is clearly the same as the marginal discounted proper density of the deficit  $|U_T|$  when  $u = 0$ . Thus, (15) may be written as

$$m_\delta(u) = \phi_\delta \int_0^u m_\delta(u-y) f_\delta(y) dy + v_\delta(u). \tag{19}$$

Clearly, (19) is a defective renewal equation, and so the generalized Gerber–Shiu function (2) satisfies a defective renewal equation which only depends on the joint distribution of  $U_{T-}$ ,  $|U_T|$ , and  $R_{N_T-1}$ .

The form of  $v_\delta(u)$  and hence also (19) simplifies in some special cases. First, if  $w(x, y, z, v) = w_{123}(x, y, z)$  so that (2) does not involve  $R_{N_T-1}$ , then the right-hand side of (12) simplifies to

$$\int_u^\infty \int_0^\infty w_{123}(x+u, y-u, u) \times \left\{ h_{1,\delta}^*(x, y | 0) + \int_0^x h_{2,\delta}^*(x, y, v | 0) dv \right\} dx dy.$$

Thus, using (14), (4) satisfies the simpler defective renewal equation

$$m_{\delta,123}(u) = \phi_\delta \int_0^u m_{\delta,123}(u-y) f_\delta(y) dy + v_{\delta,123}(u), \tag{20}$$

where

$$v_{\delta,123}(u) = \int_u^\infty \int_0^\infty w_{123}(x+u, y-u, u) h_\delta(x, y | 0) dx dy. \tag{21}$$

The special case (20) of (19) is analytically simpler due to the fact that it only involves  $h_\delta(x, y | 0)$ .

Further simplification of (21) and hence (20) occurs if  $w(x, y, z, v) = w_{23}(y, z)$ , so that only  $|U_T|$  and  $X_T$  are involved. Clearly from (18) and (21), (5) satisfies the simpler defective renewal equation

$$m_{\delta,23}(u) = \phi_\delta \int_0^u m_{\delta,23}(u-y) f_\delta(y) dy + v_{\delta,23}(u), \tag{22}$$

where

$$v_{\delta,23}(u) = \phi_\delta \int_u^\infty w_{23}(y-u, u) f_\delta(y) dy, \tag{23}$$

and it is clear from (23) that  $m_{\delta,23}(u)$  depends only on the ladder height density  $f_\delta(y)$ . Interestingly, the distribution of the last ladder height  $X_T + |U_T|$  may be determined from that of the generic ladder height distribution.

Next, we note that if  $w(x, y, z, v) = w_2(y)$ , then from (22) and (23), (6) satisfies the simpler defective renewal equation

$$m_{\delta,2}(u) = \phi_\delta \int_0^u m_{\delta,2}(u-y) f_\delta(y) dy + \phi_\delta \int_u^\infty w_2(y-u) f_\delta(y) dy. \tag{24}$$

Eq. (24) is the same defective renewal equation as in the independence case (see Willmot (2007, Eq. 2.11)), but with  $\phi_\delta$  and  $f_\delta(y)$  defined by (17) and (18) respectively. Furthermore, with  $w(x, y, z, v) = w_2(y) = 1$ , (7) satisfies

$$\bar{G}_\delta(u) = \phi_\delta \int_0^u \bar{G}_\delta(u-y) f_\delta(y) dy + \phi_\delta \bar{F}_\delta(u), \tag{25}$$

and therefore  $\bar{G}_\delta(u) = 1 - G_\delta(u)$  is (as the solution to (25) is well known to be) a compound geometric tail, i.e.

$$\bar{G}_\delta(u) = \sum_{n=1}^\infty (1 - \phi_\delta) (\phi_\delta)^n \bar{F}_\delta^{*n}(u), \quad u \geq 0,$$

where  $F_\delta(u) = 1 - \bar{F}_\delta(u) = \int_0^u f_\delta(y) dy$  and  $1 - \bar{F}_\delta^{*n}(u)$  is the distribution function of the  $n$ -fold convolution. Of course,  $\phi_\delta = \bar{G}_\delta(0)$ , and the ruin probability is given by  $\psi(u) = \bar{G}_0(u) = \Pr(T < \infty | U_0 = u)$ .

The general solution to (19) (or the special cases (20), (22) or (24)) is expressible in terms of the compound geometric density  $g_\delta(u) = -\bar{G}_\delta'(u)$  given by

$$g_\delta(u) = \sum_{n=1}^\infty (1 - \phi_\delta) (\phi_\delta)^n f_\delta^{*n}(u), \quad u \geq 0,$$

where  $f_\delta^{*n}(u) = -\frac{d}{du} \bar{F}_\delta^{*n}(u)$  is the density of the  $n$ -fold convolution of  $f_\delta(u)$ . It is well known (e.g. Resnick (1992, Section 3.5)) that

$$m_\delta(u) = v_\delta(u) + \frac{1}{1 - \phi_\delta} \int_0^u g_\delta(u-y) v_\delta(y) dy. \tag{26}$$

An alternative form of the solution which is convenient if  $v_\delta(u)$  is differentiable is (e.g. Willmot and Lin (2001, p. 154))

$$m_\delta(u) = \frac{1}{1 - \phi_\delta} \left[ v_\delta(u) - v_\delta(0)\bar{G}_\delta(u) - \int_0^u \bar{G}_\delta(u - y)v'_\delta(y)dy \right]. \tag{27}$$

For example, for the last ladder height  $X_T + |U_T|$ , the function

$$m_{\delta,5}(u) = E \left[ e^{-\delta T} w_5(X_T + |U_T|) 1(T < \infty) | U_0 = u \right]$$

satisfies, using (23)

$$m_{\delta,5}(u) = \phi_\delta \int_0^u m_{\delta,5}(u - y)f_\delta(y)dy + \phi_\delta \int_u^\infty w_5(y)f_\delta(y)dy \tag{28}$$

with solution, using (27)

$$m_{\delta,5}(u) = \frac{\phi_\delta}{1 - \phi_\delta} \left[ (1 - \bar{G}_\delta(u)) \int_u^\infty w_5(y)f_\delta(y)dy + \int_0^u (\bar{G}_\delta(u - y) - \bar{G}_\delta(u)) w_5(y)f_\delta(y)dy \right]. \tag{29}$$

As for the deficit itself, we remark that because (24) is functionally of the same form as in the more common independent case, it follows that any properties of the distribution of the deficit  $|U_T|$  are formally the same as in the independent case, but with the present definitions of  $\phi_\delta$  and  $f_\delta(y)$ . In particular, it follows directly from Willmot (2002) that

$$Pr(|U_T| > y | T < \infty) = \frac{\int_0^u \left\{ \frac{\bar{F}_0(y+u-t)}{\bar{F}_0(u-t)} \right\} \bar{F}_0(u-t) dG_0(t)}{\int_0^u \bar{F}_0(u-t) dG_0(t)},$$

so that the conditional distribution of  $|U_T|$  given that  $T < \infty$  remains a mixture of the residual lifetime distribution associated with  $F_0$ . Similarly, for the moments of the deficit, let

$$r_{k,\delta}(u) = E \left[ e^{-\delta T} (|U_T|)^k 1(T < \infty) | U_0 = u \right], \quad k = 0, 1, 2, \dots,$$

and from Willmot (2007), one has

$$r_{k,\delta}(u) = \frac{\phi_\delta}{1 - \phi_\delta} \left\{ \frac{1}{\phi_\delta} \int_u^\infty (t - u)^k dG_\delta(t) - \sum_{j=0}^k \binom{k}{j} \mu_{k-j,\delta} \int_u^\infty (t - u)^j dG_\delta(t) \right\},$$

where  $\mu_{k,\delta} = \int_0^\infty y^k dF_\delta(y)$ .

It is instructive to note that while the arguments of this section provide insight into the mathematical structure of the Gerber–Shiu functions, they do not yield information about their relationships to the claim size or interclaim time distributions (except for (8), of course). Specifically, it is usually desirable to express  $\phi_\delta$ , the ladder height density  $f_\delta(y)$ , and the discounted density  $h_\delta(x, y|0)$  in terms of quantities related to the claim size distribution  $P_f(y)$  and/or the interclaim time distribution  $K(t)$ . A commonly used approach to obtain such information is to condition on the time and the amount of the first claim (discussed further in Section 4), and then to make additional assumptions (usually about  $K(t)$ ) in order to derive either an integral or an integro-differential equation satisfied by  $m_\delta(u)$  or one of its special cases, which may then be re-expressed analytically in the form of a defective renewal equation (e.g. Li and Garrido (2005), or Boudreault et al. (2006)). The form of the defective renewal equation given in this section provides guidance insofar as the identification of these functions is concerned.

In particular, the identification of  $\phi_\delta$  and  $f_\delta(y)$  is normally easiest by using this approach with  $w(x, y, z, v) = 1$  to identify  $\bar{G}_\delta(u)$  together with (25). Such an identification then allows (22) and (24)

to be solved. The discounted density  $h_\delta(x, y|0)$  may also be identified in this manner, together with the defective renewal equation for  $m_{\delta,12}(u)$  (i.e., (20) with  $w_{123}(x, y, z) = w_{12}(x, y)$  in (21)), and (16). Once identified, (20) may be solved in full generality for  $m_{\delta,123}(u)$ . It is instructive to note that Gerber–Shiu functions involving  $X_T$  are normally not easily obtainable directly by conditioning on the time and amount of the first claim, necessitating the use of an approach such as that described here. Essentially the same approach may be also used with arbitrary  $w(x, y, z, v)$  in (19). Under different dependency assumptions (see Section 1), we remark that the identification of  $\phi_\delta$ ,  $f_\delta(y)$  and  $h_\delta(x, y|0)$  has been done in Boudreault et al. (2006, Theorem 5 and Section 5), while Cossette et al. (2008, Proposition 6) identified  $\phi_\delta$  and  $f_\delta(y)$ . We also refer the interested reader to Dickson and Hipp (2001), Gerber and Shiu (1998, 2005) and Li and Garrido (2004, 2005) for the identification of the above-mentioned related quantities in some independent Sparre Andersen risk models.

In the next section we derive the joint defective distributions of  $(U_{T-}, |U_T|, X_T, R_{N_T-1})$  and  $(U_{T-}, |U_T|, X_T)$ , as well as the marginal defective distribution of the last ladder height before ruin,  $X_T + |U_T|$ .

### 3. Associated defective distributions

We will now express the joint discounted distribution of  $(U_{T-}, |U_T|, X_T, R_{N_T-1})$  in terms of the discounted densities  $h_{1,\delta}^*(x, y|u)$  and  $h_{2,\delta}^*(x, y, v|u)$  defined in (10) and (11) respectively. We first consider the penalty function  $w(x, y, z, v) = w_{124}(x, y, v) = e^{-s_1x - s_2y - s_4v}$  as in (3), and note that in this case (12) becomes

$$v_{\delta,124}(u) = \int_u^\infty \int_0^\infty e^{-s_1(x+u) - s_2(y-u) - s_4u} h_{1,\delta}^*(x, y|0) dx dy + \int_u^\infty \int_0^\infty \int_0^x e^{-s_1(x+u) - s_2(y-u) - s_4(v+u)} \times h_{2,\delta}^*(x, y, v|0) dv dx dy.$$

Changing variables of integration yields

$$v_{\delta,124}(u) = \int_0^\infty \int_u^\infty e^{-s_1x - s_2y - s_4u} h_{1,\delta}^*(x - u, y + u|0) dx dy + \int_0^\infty \int_u^\infty \int_u^x e^{-s_1x - s_2y - s_4v} \times h_{2,\delta}^*(x - u, y + u, v - u|0) dv dx dy. \tag{30}$$

Next consider the more general penalty function  $w(x, y, z, v) = e^{-s_1x - s_2y - s_3z - s_4v}$ . With this choice of penalty function, (12) becomes  $v_\delta(u) = e^{-s_3u} v_{\delta,124}(u)$  with  $v_{\delta,124}(u)$  given by (30). Thus the Gerber–Shiu function

$$m_\delta(u) = E \left[ e^{-\delta T - s_1U_{T-} - s_2|U_T| - s_3X_T - s_4R_{N_T-1}} 1(T < \infty) | U_0 = u \right]$$

satisfies, from (26)

$$m_\delta(u) = e^{-s_3u} v_{\delta,124}(u) + \int_0^u e^{-s_3z} v_{\delta,124}(z) \frac{g_\delta(u - z)}{1 - \phi_\delta} dz$$

which may be expressed using (30) as

$$m_\delta(u) = \int_0^\infty \int_u^\infty e^{-s_1x - s_2y - s_3u - s_4u} h_{1,\delta}^*(x - u, y + u|0) dx dy + \int_0^\infty \int_u^\infty \int_u^x e^{-s_1x - s_2y - s_3u - s_4v} \times h_{2,\delta}^*(x - u, y + u, v - u|0) dv dx dy + \int_0^u \int_0^\infty \int_z^\infty e^{-s_1x - s_2y - s_3z - s_4z}$$

$$\begin{aligned} & \times \left\{ h_{1,\delta}^*(x-z, y+z|0) \frac{g_\delta(u-z)}{1-\phi_\delta} \right\} dx dy dz \\ & + \int_0^u \int_0^\infty \int_z^\infty \int_z^x e^{-s_1x-s_2y-s_3z-s_4v} \\ & \times \left\{ h_{2,\delta}^*(x-z, y+z, v-z|0) \frac{g_\delta(u-z)}{1-\phi_\delta} \right\} dv dx dy dz. \end{aligned}$$

Therefore, by the uniqueness of the Laplace–Stieltjes transform,  $(U_{T-}, |U_T|, X_T, R_{N_T-1})$  has discounted defective densities on subspaces of  $\mathbb{R}^4$  given by:

1.  $h_{12,\delta}^{**}(x, y|u) = h_{1,\delta}^*(x-u, y+u|0)$  on  $\{(x, y, z, v)|x > u, y > 0, z = u, v = u\}$  corresponding to ruin on the first claim,
2.  $h_{124,\delta}^{**}(x, y, v|u) = h_{2,\delta}^*(x-u, y+u, v-u|0)$  on  $\{(x, y, z, v)|x > u, y > 0, z = u, u < v < x\}$  corresponding to ruin on the first drop in surplus due to ruin on claims other than the first claim,
3.  $h_{123,\delta}^{**}(x, y, z|u) = h_{1,\delta}^*(x-z, y+z|0)g_\delta(u-z)/(1-\phi_\delta)$  on  $\{(x, y, z, v)|x > z, y > 0, 0 < z < u, v = z\}$  corresponding to a drop in surplus not causing ruin followed by ruin on the next claim, and
4.  $h_\delta^{**}(x, y, z, v|u) = h_{2,\delta}^*(x-z, y+z, v-z|0)g_\delta(u-z)/(1-\phi_\delta)$  on  $\{(x, y, z, v)|z < v < x, y > 0, 0 < z < u\}$  corresponding to a drop in surplus not causing ruin, followed by ruin occurring, but not on the next claim after the drop.

While it is possible to give probabilistic interpretations for the above four cases, we would like to comment on the quantity  $h_{123,\delta}^{**}(x, y, z|u)$  in detail. Note that  $g_\delta(u-z)/(1-\phi_\delta)$  can be expressed as  $\sum_{n=1}^\infty (\phi_\delta)^n f_\delta^{*n}(u-z)$ , and this can indeed be interpreted as the density for the surplus process, beginning with initial surplus  $u$ , being at level  $z$  after an arbitrary number of drops. Since the level  $z$  has to be the minimum level before ruin, the next drop (starting with level  $z$ ) has to cause ruin and this is represented by the term  $h_{1,\delta}^*(x, y|z)$ . A similar interpretation can also be given to the quantity  $h_\delta^{**}(x, y, z, v|u)$ .

We now turn to the joint discounted defective density of  $(U_{T-}, |U_T|, X_T)$ . Using the same approach with the penalty function  $w_{123}(x, y, z) = e^{-s_1x-s_2y-s_3z}$ , (21) becomes

$$\begin{aligned} v_{\delta,123}(u) &= \int_u^\infty \int_0^\infty e^{-s_1(x+u)-s_2(y-u)-s_3u} h_\delta(x, y|0) dx dy \\ &= \int_0^\infty \int_u^\infty e^{-s_1x-s_2y-s_3u} h_\delta(x-u, y+u|0) dx dy, \end{aligned}$$

and then from (20) and (26)

$$\begin{aligned} m_\delta(u) &= E \left[ e^{-\delta T-s_1U_{T-}-s_2|U_T|-s_3X_T} 1(T < \infty) | U_0 = u \right] \\ &= v_{\delta,123}(u) + \frac{1}{1-\phi_\delta} \int_0^u g_\delta(u-z) v_{\delta,123}(z) dz \\ &= \int_0^\infty \int_u^\infty e^{-s_1x-s_2y-s_3u} h_\delta(x-u, y+u|0) dx dy \\ &+ \int_0^u \int_0^\infty \int_z^\infty e^{-s_1x-s_2y-s_3z} \\ &\times \left\{ h_\delta(x-z, y+z|0) \frac{g_\delta(u-z)}{1-\phi_\delta} \right\} dx dy dz. \end{aligned}$$

Thus, by the uniqueness of the Laplace transform,  $(U_{T-}, |U_T|, X_T)$  has discounted defective densities on subspaces of  $\mathbb{R}^3$  given by:

1.  $h_{12,\delta}^{***}(x, y|u) = h_\delta(x-u, y+u|0)$  on  $\{(x, y, z)|x > u, y > 0, z = u\}$  corresponding to ruin on a first drop in surplus below  $u$ , and
2.  $h_{123,\delta}^{***}(x, y, z|u) = h_\delta(x-z, y+z|0)g_\delta(u-z)/(1-\phi_\delta)$  on  $\{(x, y, z)|x > z, y > 0, 0 < z < u\}$  corresponding to ruin occurring, but not on the first drop in surplus.

For the last ladder height before ruin  $X_T + |U_T|$ , the Laplace transform of the discounted density is given by (29) with  $w_5(y) = e^{-s_5y}$  and therefore  $X_T + |U_T|$  has the defective discounted density (given  $U_0 = u$ )

$$f_\delta(u, y) = \begin{cases} \frac{\phi_\delta}{1-\phi_\delta} [\bar{G}_\delta(u-y) - \bar{G}_\delta(u)] f_\delta(y), & y < u \\ \frac{\phi_\delta}{1-\phi_\delta} [1 - \bar{G}_\delta(u)] f_\delta(y), & y > u. \end{cases} \quad (31)$$

Note that with  $\delta = 0$  in the classical compound Poisson model without dependency (i.e.  $k(t) = \lambda e^{-\lambda t}$  and  $p_t(y) = p(y)$ ),  $h_0(x, y|0)$  in (14) equals  $(\lambda/c)p(x+y)$  (e.g. Gerber and Shiu (1997)). Thus,  $v_{0,123}(u)$  in (21) becomes the same function with a different choice of the penalty function, namely  $w_{123}(x, y, z) = w_1(x) = e^{-sx}$  and  $w_{123}(x, y, z) = w_{23}(y, z) = e^{-s(y+z)}$ . Therefore, in this case the defective density of the last ladder height before ruin given by (31) is equivalent to the defective density of the surplus prior to ruin.

The proper survival function of  $X_T + |U_T|$  given that ruin occurs is given by

$$\bar{F}_u^*(y) = \frac{\int_y^\infty f_0(u, x) dx}{\psi(u)}.$$

For  $y \geq u$ , (31) yields

$$\bar{F}_u^*(y) = \frac{\phi_0}{1-\phi_0} \left\{ \frac{1-\psi(u)}{\psi(u)} \right\} \bar{F}_0(y). \quad (32)$$

But (28) with  $\delta = 0$  and  $w_5(y) = 1$  is the (well-known) defective renewal equation for  $\psi(u)$ , namely

$$\psi(u) = \phi_0 \int_0^u \psi(u-y) f_0(y) dy + \phi_0 \bar{F}_0(u),$$

whose solution may be expressed using (26) as

$$\psi(u) = \phi_0 \bar{F}_0(u) + \frac{\phi_0}{1-\phi_0} \int_0^u \{-\psi'(u-x)\} \bar{F}_0(x) dx. \quad (33)$$

Then from (33),

$$\begin{aligned} \psi(u) &\leq \phi_0 + \frac{\phi_0}{1-\phi_0} \int_0^u \{-\psi'(u-x)\} dx \\ &= \phi_0 + \frac{\phi_0}{1-\phi_0} \{\psi(0) - \psi(u)\} \\ &= \frac{\phi_0}{1-\phi_0} \{1 - \psi(u)\} \end{aligned}$$

because  $\psi(0) = \phi_0$ . Therefore, from (32),  $\bar{F}_u^*(y) \geq \bar{F}_0(y)$ . For  $y < u$ , we obtain  $\bar{F}_u^*(y)$  as given in Box I, by integration by parts. But  $\psi(u-y) - \psi(u) = \int_0^y [-\psi'(u-x)] dx$ , and thus with  $\phi_0 = \psi(0)$  and (33),  $\bar{F}_u^*(y)$  satisfies the inequality given in Box II.

Thus,  $\bar{F}_u^*(y) \geq \bar{F}_0(y)$  for  $y \geq 0$  which implies that the last ladder height before ruin is stochastically larger than the other ladder height, in agreement with intuition.

#### 4. The adjustment coefficient and Lundberg's fundamental equation

In these generalized Sparre Andersen models, various other analytic properties hold as a consequence of the structural properties derived in Section 2. We begin with a discussion of the adjustment coefficient.

Suppose that there exists  $\kappa_\delta > 0$  satisfying

$$\int_0^\infty e^{\kappa_\delta y} f_\delta(y) dy = \frac{1}{\phi_\delta}. \quad (34)$$

$$\begin{aligned} \bar{F}_u^*(y) &= \frac{\frac{\phi_0}{1-\phi_0} \left\{ \int_y^u [\psi(u-x) - \psi(u)] f_0(x) dx + \bar{F}_0(u)[1 - \psi(u)] \right\}}{\psi(u)} \\ &= \frac{\frac{\phi_0}{1-\phi_0} \left\{ \bar{F}_0(y)[\psi(u-y) - \psi(u)] + \bar{F}_0(u)[1 - \psi(0)] + \int_y^u [-\psi'(u-x)] \bar{F}_0(x) dx \right\}}{\psi(u)} \end{aligned}$$

Box I.

$$\begin{aligned} \bar{F}_u^*(y) &= \frac{\phi_0 \bar{F}_0(u) + \frac{\phi_0}{1-\phi_0} \left\{ \bar{F}_0(y) \int_0^y [-\psi'(u-x)] dx + \int_y^u [-\psi'(u-x)] \bar{F}_0(x) dx \right\}}{\psi(u)} \\ &\geq \bar{F}_0(y) \frac{\phi_0 \bar{F}_0(u) + \frac{\phi_0}{1-\phi_0} \left\{ \int_0^y [-\psi'(u-x)] \bar{F}_0(x) dx + \int_y^u [-\psi'(u-x)] \bar{F}_0(x) dx \right\}}{\psi(u)} \\ &= \bar{F}_0(y). \end{aligned}$$

Box II.

Then, because  $\bar{G}_\delta(y)$  is a compound geometric tail, it is well known (e.g. Willmot and Lin (2001)) that (using the notation  $a(x) \sim b(x)$ ,  $x \rightarrow \infty$ , to mean  $\lim_{x \rightarrow \infty} a(x)/b(x) = 1$ )

$$\bar{G}_\delta(u) \sim C_\delta e^{-\kappa_\delta u}, \quad u \rightarrow \infty,$$

where

$$C_\delta = \frac{1 - \phi_\delta}{\phi_\delta \kappa_\delta \int_0^\infty y e^{\kappa_\delta y} f_\delta(y) dy},$$

and that

$$\bar{G}_\delta(u) \leq e^{-\kappa_\delta u}, \quad u \geq 0. \tag{35}$$

If

$$\tilde{p}_t(-\kappa_\delta) = \int_0^\infty e^{\kappa_\delta y} dP_t(y) < \infty, \tag{36}$$

then obviously  $\lim_{u \rightarrow \infty} e^{\kappa_\delta u} \bar{P}_t(u) = 0$ . Also, as (35) holds, by dominated convergence it follows that

$$\begin{aligned} \lim_{u \rightarrow \infty} e^{\kappa_\delta u} \int_0^u \bar{G}_\delta(u-y) dP_t(y) &= \int_0^\infty \left\{ \lim_{u \rightarrow \infty} e^{\kappa_\delta(u-y)} \bar{G}_\delta(u-y) \right\} e^{\kappa_\delta y} dP_t(y) \\ &= C_\delta \tilde{p}_t(-\kappa_\delta), \end{aligned}$$

and therefore

$$\lim_{u \rightarrow \infty} e^{\kappa_\delta u} \overline{G_\delta * P_t}(u) = C_\delta \tilde{p}_t(-\kappa_\delta), \tag{37}$$

where

$$\overline{G_\delta * P_t}(u) = \bar{P}_t(u) + \int_0^u \bar{G}_\delta(u-y) dP_t(y).$$

But by conditioning on the time and the amount of the first claim, one obtains

$$\bar{G}_\delta(u) = \int_0^\infty e^{-\delta t} \overline{G_\delta * P_t}(u+ct) dK(t). \tag{38}$$

Because (37) holds and  $\overline{G_\delta * P_t}(u) \leq 1$ , it follows that  $e^{\kappa_\delta u} \overline{G_\delta * P_t}(u)$  is a bounded function of  $u$  on  $(0, \infty)$ . Therefore, again using dominated convergence and (38),

$$\begin{aligned} \lim_{u \rightarrow \infty} e^{\kappa_\delta u} \bar{G}_\delta(u) &= \int_0^\infty e^{-(\delta + \kappa_\delta)t} \left\{ \lim_{u \rightarrow \infty} e^{\kappa_\delta(u+ct)} \overline{G_\delta * P_t}(u+ct) \right\} dK(t), \end{aligned}$$

i.e.

$$C_\delta = \int_0^\infty e^{-(\delta + \kappa_\delta)t} \{C_\delta \tilde{p}_t(-\kappa_\delta)\} dK(t),$$

which in turn implies that  $\kappa_\delta$  satisfies

$$E[e^{\kappa_\delta Y - (\delta + \kappa_\delta)V}] = \int_0^\infty e^{-(\delta + \kappa_\delta)t} \tilde{p}_t(-\kappa_\delta) dK(t) = 1. \tag{39}$$

To summarize, if  $\kappa_\delta$  satisfies (34) and (36) holds, then  $\kappa_\delta$  also satisfies (39), normally a more convenient relationship. Even for this fairly general model, one obtains the relative simple upper bound (35) for  $\bar{G}_\delta(u)$  (and hence also for the ruin probability by letting  $\delta = 0$ ) with  $\kappa_\delta$  obtainable from (39).

We remark that Lundberg's fundamental equation is given by

$$E[e^{-sY - (\delta - cs)V}] = 1, \tag{40}$$

and it is clear from (39) and (40) that  $s = -\kappa_\delta$  is a root of Lundberg's fundamental equation. This equation can be expected to play an integral role in the Gerber-Shiu analysis in the present model or any of its special cases, as we now demonstrate.

First, consider the function

$$\eta(u) = \int_0^\infty e^{-\delta t} \omega_t(u+ct) dK(t), \tag{41}$$

for some function  $\omega_t$ . The Laplace transform is (using a ' $\sim$ ' above the function to denote its Laplace transform)

$$\begin{aligned} \tilde{\eta}(s) &= \int_0^\infty e^{-su} \int_0^\infty e^{-\delta t} \omega_t(u+ct) dK(t) du \\ &= \int_0^\infty e^{-(\delta - cs)t} \left\{ \int_0^\infty e^{-s(u+ct)} \omega_t(u+ct) du \right\} dK(t) \\ &= \int_0^\infty e^{-(\delta - cs)t} \left\{ \tilde{\omega}_t(s) - \int_0^{ct} e^{-sx} \omega_t(x) dx \right\} dK(t) \\ &= \int_0^\infty e^{-(\delta - cs)t} \tilde{\omega}_t(s) dK(t) \\ &\quad - \int_0^\infty \int_0^{ct} e^{-\frac{1}{c} \{\delta x + (\delta - cs)(ct-x)\}} \omega_t(x) dx dK(t). \end{aligned}$$

That is,

$$\tilde{\eta}(s) = \int_0^\infty e^{-(\delta - cs)t} \tilde{\omega}_t(s) dK(t) - \tilde{\omega}_*(\delta - cs), \tag{42}$$

where

$$\tilde{\omega}_s(s) = \int_0^\infty \int_0^{ct} e^{-\frac{1}{c}\{\delta x+s(ct-x)\}} \omega_t(x) dx dK(t).$$

In order to analyze  $m_\delta(u)$ , it is sufficient by the results of Section 2 to consider the special case  $m_{\delta,124}(u)$ . By conditioning on the time and the amount of the first claim, it follows that  $m_{\delta,124}(u)$  satisfies the integral equation

$$m_{\delta,124}(u) = \beta_\delta(u) + \int_0^\infty e^{-\delta t} \sigma_{t,\delta}(u+ct) dK(t), \quad (43)$$

where

$$\sigma_{t,\delta}(x) = \int_0^x m_{\delta,124}(x-y) dP_t(y), \quad (44)$$

and

$$\beta_\delta(u) = \int_0^\infty e^{-\delta t} \int_{u+ct}^\infty w_{124}(u+ct, y-u-ct, u) dP_t(y) dK(t). \quad (45)$$

We remark that (38) is the special case of (43) with  $w_{124}(x, y, v) = 1$ . Also,  $\beta_\delta(u)$  is the contribution to the penalty function due to ruin on the first claim, as is clear from the alternative representation (easily established by changing the variables of integration)

$$\beta_\delta(u) = \int_u^\infty \int_0^\infty e^{-\delta(\frac{x-u}{c})} w_{124}(x, y, u) h_1^*(x, y|u) dy dx.$$

The term on the right-hand side of (43) is of the form (41), and thus taking Laplace transforms of (43) yields, using (42)

$$\tilde{m}_{\delta,124}(s) = \tilde{\beta}_\delta(s) + \int_0^\infty e^{-(\delta-cs)t} \tilde{\sigma}_{t,\delta}(s) dK(t) - \tilde{\sigma}_w(\delta-cs), \quad (46)$$

where

$$\tilde{\sigma}_w(s) = \int_0^\infty \int_0^{ct} e^{-\frac{1}{c}\{\delta x+s(ct-x)\}} \sigma_{t,\delta}(x) dx dK(t).$$

But  $\tilde{\sigma}_{t,\delta}(s) = \tilde{m}_{\delta,124}(s) \tilde{p}_t(s)$  from (44), and thus (46) may be expressed as

$$\begin{aligned} \tilde{m}_{\delta,124}(s) &= \tilde{\beta}_\delta(s) + \tilde{m}_{\delta,124}(s) \int_0^\infty e^{-(\delta-cs)t} \tilde{p}_t(s) dK(t) - \tilde{\sigma}_w(\delta-cs), \end{aligned}$$

and because  $E[e^{-sY-(\delta-cs)V}] = \int_0^\infty e^{-(\delta-cs)t} \tilde{p}_t(s) dK(t)$ , it follows that

$$\{1 - E[e^{-sY-(\delta-cs)V}]\} \tilde{m}_{\delta,124}(s) = \tilde{\beta}_\delta(s) - \tilde{\sigma}_w(\delta-cs). \quad (47)$$

Note that the left side of (47) is 0 if  $s$  is replaced by a root (with nonnegative real part) of Lundberg's generalized equation (40). This allows for identification of unknown quantities in the term  $\tilde{\sigma}_w(\delta-cs)$  on the right side of (47), a step generally needed to ultimately invert (either numerically or analytically under some additional conditions on the distributions of the interclaim time  $V$  and/or the claim size  $Y$ ) the Laplace transform  $\tilde{m}_{\delta,124}(s)$ .

In order to examine further the structure of the defective renewal equation

$$m_{\delta,124}(u) = \phi_\delta \int_0^u m_{\delta,124}(u-y) f_\delta(y) dy + v_{\delta,124}(u), \quad (48)$$

consider the special case  $w_{124}(x, y, v) = 1$ , in which case  $m_\delta(u)$  reduces to  $\bar{G}_\delta(u)$  given by (7). In this case from (45),  $\beta_\delta(u)$  becomes  $\int_0^\infty e^{-\delta t} \bar{P}_t(u+ct) dK(t)$ , which is again of the form (41), and therefore with  $w_{124}(x, y, v) = 1$ , (47) becomes

$$\begin{aligned} \{1 - E[e^{-sY-(\delta-cs)V}]\} \tilde{\bar{G}}_\delta(s) &= \int_0^\infty e^{-(\delta-cs)t} \left\{ \frac{1 - \tilde{p}_t(s)}{s} \right\} dK(t) - \tilde{\varphi}(\delta-cs), \end{aligned}$$

where

$$\begin{aligned} \tilde{\varphi}(s) &= \int_0^\infty \int_0^{ct} e^{-\frac{1}{c}\{\delta x+s(ct-x)\}} \\ &\quad \times \left\{ \bar{P}_t(x) + \int_0^x \bar{G}_\delta(x-y) dP_t(y) \right\} dK(t). \end{aligned}$$

That is,

$$\begin{aligned} \{1 - E[e^{-sY-(\delta-cs)V}]\} \tilde{\bar{G}}_\delta(s) &= \frac{\tilde{k}(\delta-cs) - E[e^{-sY-(\delta-cs)V}]}{s} - \tilde{\varphi}(\delta-cs), \end{aligned}$$

and again unknown constants in the function  $\tilde{\varphi}(\delta-cs)$  may typically be identified in terms of roots of Lundberg's generalized equation (40). Thus,

$$\begin{aligned} s \tilde{\bar{G}}_\delta(s) &= \frac{\tilde{k}(\delta-cs) - E[e^{-sY-(\delta-cs)V}] - s \tilde{\varphi}(\delta-cs)}{1 - E[e^{-sY-(\delta-cs)V}]} \\ &= 1 - \frac{1 - \tilde{k}(\delta-cs) + s \tilde{\varphi}(\delta-cs)}{1 - E[e^{-sY-(\delta-cs)V}]}. \end{aligned} \quad (49)$$

On the other hand, taking Laplace transforms of (25) results in

$$s \tilde{\bar{G}}_\delta(s) = \frac{\phi_\delta \{1 - \tilde{f}_\delta(s)\}}{1 - \phi_\delta \tilde{f}_\delta(s)} = 1 - \frac{1 - \phi_\delta}{1 - \phi_\delta \tilde{f}_\delta(s)}, \quad (50)$$

and equating the right-hand side of (49) to that of (50) results in an expression for the compound geometric Laplace transform, namely

$$\frac{1 - \phi_\delta}{1 - \phi_\delta \tilde{f}_\delta(s)} = \frac{1 - \tilde{k}(\delta-cs) + s \tilde{\varphi}(\delta-cs)}{1 - E[e^{-sY-(\delta-cs)V}]}. \quad (51)$$

Eq. (51) may be re-expressed as

$$\begin{aligned} 1 - E[e^{-sY-(\delta-cs)V}] &= \frac{1 - \tilde{k}(\delta-cs) + s \tilde{\varphi}(\delta-cs)}{1 - \phi_\delta} \{1 - \phi_\delta \tilde{f}_\delta(s)\}, \end{aligned} \quad (52)$$

and substitution of (52) into (47) results in

$$\{1 - \phi_\delta \tilde{f}_\delta(s)\} \tilde{m}_{\delta,124}(s) = \tilde{v}_{\delta,124}(s),$$

where

$$\tilde{v}_{\delta,124}(s) = \frac{(1 - \phi_\delta) \{\tilde{\beta}_\delta(s) - \tilde{\sigma}_w(\delta-cs)\}}{1 - \tilde{k}(\delta-cs) + s \tilde{\varphi}(\delta-cs)},$$

and inversion yields (48). As shown in Section 2, analysis of the more general  $m_\delta(u)$  defined by (2) is possible using these results for  $m_{\delta,124}(u)$ .

Also,  $\phi_\delta = \bar{G}_\delta(0)$  may be obtained by taking the limit as  $s \rightarrow \infty$  of the right-hand side of (49) and using the initial value theorem.

In principle, the solution  $m_{\delta,124}(u)$  (or its special cases) may be used to identify any or all of  $h_{2,\delta}^*(x, y, v|0)$ ,  $h_\delta(x, y|0)$ , or  $f_\delta(y)$ . In turn, the Gerber–Shiu functions  $m_\delta(u)$ ,  $m_{\delta,123}(u)$ , or  $m_{\delta,23}(u)$ , respectively, may then be obtained by solving the defective renewal equations (19), (20), or (22), so that  $X_T$  may be incorporated into the analysis. In the classical compound Poisson risk model,  $m_{\delta,124}(u)$  along with  $h_{2,\delta}^*(x, y, v|0)$  have been obtained by Cheung et al. (in press), and Willmot and Woo (submitted for publication) generalized the results to a Sparre Andersen risk model with a Coxian interclaim time distribution. See also Li and Garrido (2005) for a identification of  $h_\delta(x, y|0)$  and  $f_\delta(y)$  in the latter model.

### 5. Sparre Andersen models without dependency

Simplifications do result in the independence situation, i.e. with  $\bar{P}_t(y) = \bar{P}(y)$  and  $p_t(y) = p(y)$ . As in Gerber and Shiu (1998), the conditional density of  $|U_T|$  given  $U_T = x, R_{N_T-1} = v, N_T \geq 2$ , and

$T = t$  is given by  $p(x + y) / \bar{P}(x)$ , so that one may write

$$h_2^*(t, x, y, v|u) = \frac{p(x + y)}{\bar{P}(x)} h^{(2)}(t, x, v|u), \tag{53}$$

where  $h^{(2)}(t, x, v|u)$  represents the joint defective density of  $T$ ,  $U_{T-}$  and  $R_{N_{T-1}}$  for ruin occurring on claims subsequent to the first. Therefore, from (53),

$$h_{2,\delta}^*(x, y, v|u) = \frac{p(x + y)}{\bar{P}(x)} h_\delta^{(2)}(x, v|u), \tag{54}$$

where  $h_\delta^{(2)}(x, v|u) = \int_0^\infty e^{-\delta t} h^{(2)}(t, x, v|u) dt$ . Thus, using (8), (10) and (54), the discounted density (14) may be expressed as

$$h_\delta(x, y|u) = \frac{p(x + y)}{\bar{P}(x)} h_\delta(x|u), \tag{55}$$

where

$$h_\delta(x|u) = \frac{1}{c} e^{-\delta(\frac{x-u}{c})} k\left(\frac{x-u}{c}\right) \bar{P}(x) + \int_0^x h_\delta^{(2)}(x, v|u) dv$$

is the discounted (marginal if  $\delta = 0$ ) density of the surplus prior to ruin  $U_{T-}$ .

Hence, (17) becomes, using (55),

$$\phi_\delta = \int_0^\infty h_\delta(x|0) \left\{ \int_0^\infty \frac{p(x + y)}{\bar{P}(x)} dy \right\} dx = \int_0^\infty h_\delta(x|0) dx,$$

and (18) may be expressed as the mixed density

$$f_\delta(y) = \int_0^\infty \left\{ \frac{h_\delta(x|0)}{\phi_\delta} \right\} \frac{p(x + y)}{\bar{P}(x)} dx. \tag{56}$$

The defective renewal equation may also be simplified in some cases. When  $w(x, y, z, v) = w_{123}(x, y, z)$ , substitution of (55) into (21) yields

$$\begin{aligned} v_{\delta, 123}(u) &= \int_u^\infty \int_0^\infty w_{123}(x + u, y - u, u) \left\{ \frac{p(x + y)}{\bar{P}(x)} \right\} h_\delta(x|0) dx dy \\ &= \int_0^\infty \left\{ \int_u^\infty w_{123}(x + u, y - u, u) p(x + y) dy \right\} \frac{h_\delta(x|0)}{\bar{P}(x)} dx \\ &= \int_0^\infty \left\{ \int_{x+u}^\infty w_{123}(x + u, y - x - u, u) p(y) dy \right\} \frac{h_\delta(x|0)}{\bar{P}(x)} dx. \end{aligned}$$

That is,

$$v_{\delta, 123}(u) = \int_0^\infty \alpha_{123}(x + u, u) \frac{h_\delta(x|0)}{\bar{P}(x)} dx, \tag{57}$$

where

$$\alpha_{123}(x, u) = \int_x^\infty w_{123}(x, y - x, u) p(y) dy. \tag{58}$$

The usual Gerber–Shiu function (1) thus satisfies the defective renewal equation (from (20))

$$m_{\delta, 12}(u) = \phi_\delta \int_0^u m_{\delta, 12}(u - y) f_\delta(y) dy + v_{\delta, 12}(u),$$

where, using (57) and (58),

$$v_{\delta, 12}(u) = \int_0^\infty \alpha_{12}(x + u) \frac{h_\delta(x|0)}{\bar{P}(x)} dx,$$

with

$$\alpha_{12}(x) = \int_x^\infty w_{12}(x, y - x) p(y) dy.$$

Also, if  $w(x, y, z, v) = w_{134}(x, z, v) w_2(y)$ , then (13) may be expressed using (54) as

$$\begin{aligned} v_{\delta, 134,2}(u) &= \int_0^\infty e^{-\delta t} w_{134}(u + ct, u, u) \\ &\quad \times \left\{ \int_0^\infty w_2(y) p(y + ct + u) dy \right\} k(t) dt \\ &\quad + \int_0^\infty \frac{1}{\bar{P}(x)} \left\{ \int_0^\infty w_2(y) p(x + y + u) dy \right\} \\ &\quad \times \int_0^x w_{134}(x + u, u, v + u) h_\delta^{(2)}(x, v|0) dv dx. \end{aligned} \tag{59}$$

We now illustrate some of these ideas by deriving the joint Laplace transform of all these quantities in the case with exponential claim sizes.

**Example – Exponential claim sizes and arbitrary interclaim times**

We consider the joint Laplace transform of  $(T, U_{T-}, |U_T|, X_T, R_{N_{T-1}})$  when  $p(y) = \beta e^{-\beta y}$ . Letting  $w(x, y, z, v) = e^{-s_1 x - s_2 y - s_3 z - s_4 v}$ , (59) yields

$$\begin{aligned} v_\delta(u) &= \int_0^\infty e^{-\delta t - s_1(u+ct) - s_3 u - s_4 u} \\ &\quad \times \left\{ \int_0^\infty e^{-s_2 y} \beta e^{-\beta(y+ct+u)} dy \right\} k(t) dt \\ &\quad + \int_0^\infty \left\{ \int_0^\infty e^{-s_2 y} \beta e^{-\beta(y+u)} dy \right\} \\ &\quad \times \int_0^x e^{-s_1(x+u) - s_3 u - s_4(v+u)} h_\delta^{(2)}(x, v|0) dv dx \\ &= e^{-(\beta+s_1+s_3+s_4)u} \int_0^\infty e^{-(\delta+cs_1+c\beta)t} \\ &\quad \times \left\{ \int_0^\infty \beta e^{-(\beta+s_2)y} dy \right\} k(t) dt \\ &\quad + e^{-(\beta+s_1+s_3+s_4)u} \left\{ \int_0^\infty \beta e^{-(\beta+s_2)y} dy \right\} \\ &\quad \times \int_0^\infty \int_0^x e^{-s_1 x - s_4 v} h_\delta^{(2)}(x, v|0) dv dx \\ &= \frac{\beta e^{-(\beta+s_1+s_3+s_4)u}}{\beta + s_2} \left\{ \tilde{k}(\delta + cs_1 + c\beta) + \tilde{h}_\delta^{(2)}(s_1, s_4|0) \right\}, \end{aligned}$$

where  $\tilde{h}_\delta^{(2)}(s_1, s_4|0) = \int_0^\infty \int_0^x e^{-s_1 x - s_4 v} h_\delta^{(2)}(x, v|0) dv dx$  is the (bivariate) Laplace transform of  $h_\delta^{(2)}(x, v|0)$ . For notational convenience, let  $\gamma_\delta(s_1, s_4) = \tilde{k}(\delta + cs_1 + c\beta) + \tilde{h}_\delta^{(2)}(s_1, s_4|0)$ , so that

$$v_\delta(u) = \frac{\beta \gamma_\delta(s_1, s_4)}{\beta + s_2} e^{-(\beta+s_1+s_3+s_4)u}. \tag{60}$$

It is clear from (56) that  $f_\delta(y) = \beta e^{-\beta y}$  in this case. Thus, from (19), the Gerber–Shiu function

$$m_\delta(u) = E \left[ e^{-\delta T - s_1 U_{T-} - s_2 |U_T| - s_3 X_T - s_4 R_{N_{T-1}}} \mathbf{1}(T < \infty) | U_0 = u \right]$$

satisfies

$$m_\delta(u) = \phi_\delta \int_0^u m_\delta(u - y) \beta e^{-\beta y} dy + v_\delta(u),$$

where  $v_\delta(u)$  is given by (60). To solve this equation we will use Laplace transforms. Thus,

$$\tilde{m}_\delta(z) = \phi_\delta \tilde{m}_\delta(z) \frac{\beta}{\beta + z} + \frac{\beta \gamma_\delta(s_1, s_4)}{\beta + s_2} (\beta + s_1 + s_3 + s_4 + z)^{-1},$$

and hence solving for  $\tilde{m}_\delta(z)$  yields

$$\begin{aligned} \tilde{m}_\delta(z) &= \frac{\beta\gamma_\delta(s_1, s_4)}{\beta + s_2} \frac{(\beta + s_1 + s_3 + s_4 + z)^{-1}}{1 - \phi_\delta\beta(\beta + z)^{-1}} \\ &= \frac{\beta\gamma_\delta(s_1, s_4)}{(\beta + s_2)(\phi_\delta\beta + s_1 + s_3 + s_4)} \\ &\quad \times \left\{ \frac{s_1 + s_3 + s_4}{\beta + s_1 + s_3 + s_4 + z} + \frac{\phi_\delta\beta}{\beta(1 - \phi_\delta) + z} \right\} \end{aligned}$$

after a little algebra. Thus inversion with respect to  $z$  yields

$$\begin{aligned} m_\delta(u) &= \frac{\beta\gamma_\delta(s_1, s_4)}{(\beta + s_2)(\phi_\delta\beta + s_1 + s_3 + s_4)} \\ &\quad \times \left\{ (s_1 + s_3 + s_4)e^{-(\beta+s_1+s_3+s_4)u} + \beta\bar{G}_\delta(u) \right\}, \quad (61) \end{aligned}$$

where  $\bar{G}_\delta(u) = \phi_\delta e^{-\beta(1-\phi_\delta)u}$  with  $\phi_\delta$  the solution to  $\phi_\delta = k(\delta + c\beta - \phi_\delta c\beta)$  (e.g. Willmot (2007)).

It is useful to be able to express  $\tilde{h}_\delta^{(2)}(s_1, s_4|0)$  or equivalently  $\gamma_\delta(s_1, s_4)$  in terms of the interclaim time Laplace transform  $\tilde{k}(s)$ . To do this, we will examine  $m_\delta(u)$  by conditioning on the time and amount of the first claim, which simplifies if we ignore  $X_T$  by letting  $s_3 = 0$  (and for simplicity we will also set  $s_2 = 0$ ). Thus, let

$$m_{\delta,14}(u) = \frac{\gamma_\delta(s_1, s_4)}{\phi_\delta\beta + s_1 + s_4} \left\{ (s_1 + s_4)e^{-(\beta+s_1+s_4)u} + \beta\bar{G}_\delta(u) \right\}, \quad (62)$$

which corresponds to the choice of the penalty function  $w(x, y, z, v) = w_{14}(x, v) = e^{-s_1x - s_4v}$ . Thus, in this case,  $m_{\delta,14}(u)$  satisfies the integral equation (from (43))

$$m_{\delta,14}(u) = \int_0^\infty e^{-\delta t} \tau_\delta(u + ct, u)k(t)dt, \quad (63)$$

where

$$\tau_\delta(t, u) = \int_0^t m_{\delta,14}(t - y)\beta e^{-\beta y}dy + \int_t^\infty e^{-s_1t - s_4u}\beta e^{-\beta y}dy.$$

Clearly,

$$\int_t^\infty e^{-s_1t - s_4u}\beta e^{-\beta y}dy = e^{-(\beta+s_1)t - s_4u},$$

and using (62)

$$\begin{aligned} &\int_0^t m_{\delta,14}(t - y)\beta e^{-\beta y}dy \\ &= \frac{\gamma_\delta(s_1, s_4)}{\phi_\delta\beta + s_1 + s_4} \left\{ (s_1 + s_4) \int_0^t e^{-(\beta+s_1+s_4)(t-y)}\beta e^{-\beta y}dy \right. \\ &\quad \left. + \beta \int_0^t \bar{G}_\delta(t - y)f_\delta(y)dy \right\} \\ &= \frac{\beta\gamma_\delta(s_1, s_4)}{\phi_\delta\beta + s_1 + s_4} \left\{ e^{-\beta t} [1 - e^{-(s_1+s_4)t}] + \left[ \frac{\bar{G}_\delta(t)}{\phi_\delta} - e^{-\beta t} \right] \right\} \\ &= \frac{\beta\gamma_\delta(s_1, s_4)}{\phi_\delta\beta + s_1 + s_4} \left\{ \frac{\bar{G}_\delta(t)}{\phi_\delta} - e^{-(\beta+s_1+s_4)t} \right\}. \end{aligned}$$

Thus,

$$\tau_\delta(t, u) = \frac{\beta\gamma_\delta(s_1, s_4)}{\phi_\delta\beta + s_1 + s_4} \left\{ \frac{\bar{G}_\delta(t)}{\phi_\delta} - e^{-(\beta+s_1+s_4)t} \right\} + e^{-(\beta+s_1)t - s_4u},$$

and therefore

$$\begin{aligned} \int_0^\infty e^{-\delta t} \tau_\delta(u + ct, u)k(t)dt &= \int_0^\infty e^{-\delta t - (\beta+s_1)(u+ct) - s_4u}k(t)dt \\ &+ \frac{\beta\gamma_\delta(s_1, s_4)}{\phi_\delta\beta + s_1 + s_4} \left\{ \frac{1}{\phi_\delta} \int_0^\infty e^{-\delta t} \bar{G}_\delta(u + ct)k(t)dt \right. \\ &\quad \left. - \int_0^\infty e^{-\delta t - (\beta+s_1+s_4)(u+ct)}k(t)dt \right\}. \end{aligned}$$

That is

$$\begin{aligned} \int_0^\infty e^{-\delta t} \tau_\delta(u + ct, u)k(t)dt &= e^{-(\beta+s_1+s_4)u} \tilde{k}(\delta + c\beta + cs_1) \\ &+ \frac{\beta\gamma_\delta(s_1, s_4)}{\phi_\delta\beta + s_1 + s_4} \left\{ \int_0^\infty e^{-\delta t - \beta(1-\phi_\delta)(u+ct)}k(t)dt \right. \\ &\quad \left. - e^{-(\beta+s_1+s_4)u} \tilde{k}(\delta + c\beta + cs_1 + cs_4) \right\}. \end{aligned}$$

But  $\int_0^\infty e^{-\delta t - c\beta(1-\phi_\delta)t}k(t)dt = \phi_\delta$ , and thus

$$\begin{aligned} \int_0^\infty e^{-\delta t} \tau_\delta(u + ct, u)k(t)dt &= e^{-(\beta+s_1+s_4)u} \tilde{k}(\delta + c\beta + cs_1) \\ &+ \frac{\beta\gamma_\delta(s_1, s_4)}{\phi_\delta\beta + s_1 + s_4} \left\{ \bar{G}_\delta(u) - e^{-(\beta+s_1+s_4)u} \tilde{k}(\delta + c\beta + cs_1 + cs_4) \right\}, \quad (64) \end{aligned}$$

which (by (63)) equals  $m_{\delta,14}(u)$ . Thus, equating (62) and (64), the terms involving  $\bar{G}_\delta(u)$  cancel, and division by  $e^{-(\beta+s_1+s_4)u}$  results in

$$\begin{aligned} &\frac{\gamma_\delta(s_1, s_4)}{\phi_\delta\beta + s_1 + s_4} (s_1 + s_4) \\ &= \tilde{k}(\delta + c\beta + cs_1) - \frac{\gamma_\delta(s_1, s_4)}{\phi_\delta\beta + s_1 + s_4} \beta \tilde{k}(\delta + c\beta + cs_1 + cs_4), \end{aligned}$$

which in turn implies that

$$\gamma_\delta(s_1, s_4) = \frac{(\phi_\delta\beta + s_1 + s_4)\tilde{k}(\delta + c\beta + cs_1)}{s_1 + s_4 + \beta\tilde{k}(\delta + c\beta + cs_1 + cs_4)}. \quad (65)$$

Finally, substitution of (65) into (61) yields

$$m_\delta(u) = C_\delta(s_1, s_2, s_3, s_4) \left\{ (s_1 + s_3 + s_4)e^{-(\beta+s_1+s_3+s_4)u} + \phi_\delta\beta e^{-\beta(1-\phi_\delta)u} \right\}, \quad (66)$$

where

$$\begin{aligned} C_\delta(s_1, s_2, s_3, s_4) &= \frac{\beta(\phi_\delta\beta + s_1 + s_4)\tilde{k}(\delta + c\beta + cs_1)}{(\beta + s_2)(\phi_\delta\beta + s_1 + s_3 + s_4)\{s_1 + s_4 + \beta\tilde{k}(\delta + c\beta + cs_1 + cs_4)\}}. \quad (67) \end{aligned}$$

The last interclaim time before ruin  $V_{N_T} = (U_T - R_{N_T-1})/c$  was analyzed in the classical compound Poisson risk model by Cheung et al. (in press). For the present Sparre Andersen model with exponential claims, the Laplace transform of the defective distribution of  $V_{N_T}$  is given by (66) with  $\delta = 0$ ,  $s_1 = s/c$ ,  $s_2 = s_3 = 0$ , and  $s_4 = -s/c$ .

Thus, using (67), it follows that

$$E[e^{-sV_{N_T}} 1(T < \infty) | U_0 = u] = \frac{\tilde{k}(c\beta + s)}{\tilde{k}(c\beta)} \psi(u),$$

and the proper distribution of  $V_{N_T} | T < \infty$  is functionally independent of  $u$  with Laplace transform

$$E[e^{-sV_{N_T}} | T < \infty] = \frac{\tilde{k}(c\beta + s)}{\tilde{k}(c\beta)}. \quad (68)$$

Clearly, (68) is the Laplace transform of an Esscher transformed distribution of  $K(t)$ , so that if  $K_1(t) = 1 - \bar{K}_1(t) = \Pr(V_{N_T} | T < \infty)$  is the distribution function, the density  $k_1(t) = K_1'(t)$  is given by

$$k_1(t) = \frac{e^{-c\beta t}k(t)}{\tilde{k}(c\beta)}. \quad (69)$$

The evaluation of  $k_1(t)$  is straightforward for many choices of  $k(t)$ . In particular, if  $k(t)$  is from the mixed Erlang, combination of

exponentials, or phase-type classes, the same is easily seen to be true for  $k_1(t)$ .

Also,  $V_{N_T}|T < \infty$  is stochastically dominated by the generic interclaim time random variable  $V$ , a result which agrees with intuition. To see this, note that the failure rate corresponding to  $K_1(t)$  satisfies, from (69),

$$\begin{aligned} \mu_1(t) &= -\frac{d}{dt} \ln \bar{K}_1(t) \\ &= \frac{e^{-c\beta t} k(t)}{\int_t^\infty e^{-c\beta y} k(y) dy} \\ &= \frac{k(t)}{\int_t^\infty e^{-c\beta(y-t)} k(y) dy}, \end{aligned}$$

from which because  $e^{-c\beta(y-t)} \leq 1$ , it is clear that  $\mu_1(t) \geq \mu(t)$ , where  $\mu(t) = k(t)/\bar{K}(t)$  is the failure rate of  $K(t)$ . Thus,

$$\bar{K}_1(t) = e^{-\int_0^t \mu_1(x) dx} \leq e^{-\int_0^t \mu(x) dx} = \bar{K}(t).$$

This stochastic bound may be improved upon if  $K(t)$  is from the new worse (better) than used or NWU (NBU) class of distributions for which  $\bar{K}(t+y) \geq (\leq) \bar{K}(t)\bar{K}(y)$  for all  $t \geq 0$  and  $y \geq 0$ , a class which includes distributions with a nonincreasing (nondecreasing) failure rate (e.g. Barlow and Proschan (1975)). Integration by parts yields

$$\begin{aligned} &\int_t^\infty e^{-c\beta y} k(y) dy \\ &= e^{-c\beta t} \left\{ \bar{K}(t) - c\beta \int_0^\infty e^{-c\beta y} \bar{K}(y+t) dy \right\}. \end{aligned} \tag{70}$$

Therefore, if  $K(t)$  is NWU (NBU), the use of (70) at  $t = 0$  results in

$$\begin{aligned} \int_t^\infty e^{-c\beta y} k(y) dy &\leq (\geq) e^{-c\beta t} \bar{K}(t) \left\{ 1 - c\beta \int_0^\infty e^{-c\beta y} \bar{K}(y) dy \right\} \\ &= e^{-c\beta t} \bar{K}(t) \int_0^\infty e^{-c\beta y} k(y) dy. \end{aligned}$$

In other words, if  $K(t)$  is NWU then  $\bar{K}_1(t) \leq e^{-c\beta t} \bar{K}(t)$  (an equality if  $\bar{K}(t) = e^{-\lambda t}$ ), whereas if  $K(t)$  is NBU then  $e^{-c\beta t} \bar{K}(t) \leq \bar{K}_1(t) \leq \bar{K}(t)$ . □

For more general claim size distributions, a similar approach may be used to determine the joint Laplace transform as in Landriault and Willmot (2008).

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**References**

Albrecher, Hansjoerg, Boxma, Onno J., 2004. A ruin model with dependence between claim sizes and claim intervals. *Insurance: Mathematics and Economics* 35 (2), 245–254.

Albrecher, Hansjoerg, Teugels, Jef L., 2006. Exponential behavior in the presence of dependence in risk theory. *Journal of Applied Probability* 43 (1), 257–273.

Badescu, Andrei L., Cheung, Eric C.K., Landriault, David, 2009. Dependent risk models with bivariate phase-type distributions. *Journal of Applied Probability* 46 (1), 113–131.

Barlow, Richard E., Proschan, Frank, 1975. *Statistical Theory of Reliability and Life Testing: Probability Models*. Holt, Rinehart, and Winston, New York.

Boudreault, Mathieu, Cossette, Helene, Landriault, David, Marceau, Etienne, 2006. On a risk model with dependence between interclaim arrivals and claim sizes. *Scandinavian Actuarial Journal* 5, 265–285.

Cheung, Eric C.K., Landriault, David, Willmot, Gordon E., Woo, Jae-Kyung, 2009. Gerber–Shiu analysis with a generalized penalty function. *Scandinavian Actuarial Journal* (in press).

Cossette, Helene, Marceau, Etienne, Marri, Fouad, 2008. On the compound Poisson risk model with dependence based on a generalized Farlie–Gumbel–Morgenstern copula. *Insurance: Mathematics and Economics* 43 (3), 444–455.

Doney, R.A., Kyprianou, A.E., 2006. Overshoots and undershoots of Lévy processes. *Annals of Applied Probability* 16 (1), 91–106.

Dickson, David C.M., Hipp, Christian, 2001. On the time to ruin for Erlang (2) risk processes. *Insurance: Mathematics and Economics* 29 (3), 333–344.

Dufresne, François, Gerber, Hans U., 1988. The surpluses immediately before and at ruin, and the amount of the claim causing ruin. *Insurance: Mathematics and Economics* 7 (3), 193–199.

Gerber, Hans U., Shiu, Elias S.W., 1997. The joint distribution of the time of ruin, the surplus immediately before ruin, and the deficit at ruin. *Insurance: Mathematics and Economics* 21 (2), 129–137.

Gerber, Hans U., Shiu, Elias S.W., 1998. On the time value of ruin. *North American Actuarial Journal* 2 (1), 48–78.

Gerber, Hans U., Shiu, Elias S.W., 2005. The time value of ruin in a Sparre Andersen model. *North American Actuarial Journal* 9 (2), 49–84.

Landriault, David, Willmot, Gordon E., 2008. On the Gerber–Shiu discounted penalty function in the Sparre Andersen model with an arbitrary interclaim time distribution. *Insurance: Mathematics and Economics* 42 (2), 600–608.

Li, Shuanming, Garrido, Jose, 2004. On ruin for the Erlang (n) risk process. *Insurance: Mathematics and Economics* 34 (3), 391–408.

Li, Shuanming, Garrido, Jose, 2005. On a general class of renewal risk process: Analysis of the Gerber–Shiu function. *Advances in Applied Probability* 37 (3), 836–856.

Resnick, S., 1992. *Adventures in Stochastic Processes*. Birkhauser, Boston.

Willmot, Gordon E., 2002. Compound geometric residual lifetime distributions and the deficit at ruin. *Insurance: Mathematics and Economics* 30 (3), 421–438.

Willmot, Gordon E., 2007. On the discounted penalty function in the renewal risk model with general interclaim times. *Insurance: Mathematics and Economics* 41 (1), 17–31.

Willmot, Gordon E., Lin, Sheldon X., 2001. *Lundberg Approximations for Compound Distributions with Applications*. Springer-Verlag, New York.

Willmot, Gordon E., Woo, Jae-Kyung, 2009. Generalized penalty function analysis with a class of Coxian interclaim time distribution (submitted for publication).