Optimal Risk-Sharing with Effort and Project Choice *

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Abstract

We consider first-best risk-sharing problems in which "the agent" can control both the drift (effort choice) and the volatility of the underlying process (project selection). In a model of delegated portfolio management, it is optimal to compensate the manager with an option-type payoff, where the functional form of the option is obtained as a solution to an ordinary differential equation. In the general case, the optimal contract is a fixed point of a functional that connects the agent's and the principal's maximization problems. We apply martingale/duality methods familiar from optimal consumption-investment problems.

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1 Introduction

The problem of risk-sharing, or how to split an aggregate uncertain endowment between two or more individuals, has a long tradition in Economics, starting with Arrow [2], followed by Borch [4], Wilson [41] and Amershi and Stoeckenius [1], among others. All those papers consider the problem in a static setting and link it to Pareto-optimization. An influential strand of dynamic models starts with Lucas and Stokey [28], followed by Epstein [17]. They introduce recursive utilities. Risk-sharing with several agents and recursive utilities is studied in Duffie, Geoffard and Skiadas [15] and Dumas, Uppal and Wang [16]. Dynamic optimization techniques in continuous-time are introduced by Merton [29] for a consumption/portfolio optimization problem. Technically, the choice of consumption is equivalent to the choice of effort in risk-sharing problems, while the choice of an optimal portfolio is equivalent to the choice of project. For this reason, these optimization techniques are suitable for risk-sharing problems. An early important paper in continuous-time considering a Principal-Agent problem with moral hazard is Holmström and Milgrom [20]. They consider exponential utility and show that the second-best contract is linear. That result has been extended by Schättler and Sung [37] and Sung [39] to the case in which the agent also chooses the project.[1] More recently, Müller [30, 31] finds the full information (the "first-best") risk-sharing solution of the Holmström and Milgrom [20] problem, which is linear as well, and shows how it can be approximated by control revisions taking place at discrete times. Bolton and Harris [3] consider a very general setting. Sannikov [36] considers the case of hidden information for more general types of utilities than Holmström and Milgrom [20]. In the same spirit, we mention Detemple, Govindaraj and Loewenstein [14] and Cvitanić, Wan and Zhang [11]. Williams [40] introduces the Backward Stochastic Differential Equations approach to the problems with moral hazard, and also includes a state variable, possibly unobserved by the

[1] These papers are related to a different strand of the literature that considers risk-sharing with "hidden action" in dynamic discrete-time settings: Spear and Srivastava [38] characterize the general solution, using dynamic programming principles; Phelan and Towsend [33] present a numerical algorithm to solve a general set of dynamic problems with hidden action; DeMarzo and Fishman [13] apply the dynamic programming results of the former to a large number of problems affecting the dynamics of a firm.

principal. An important recent paper is Ou-Yang [32], which considers a similar risk-sharing problem, but without independent control of the drift. His focus is on the project selection problem, applied to the choice of a portfolio by a money manager acting for an investor. Finally, we mention Larsen [27], which solves numerically the case with power utilities for the linear, portfolio delegation case, for contracts which depend only on the final value of the portfolio.

Most of the continuous-time papers use the techniques first introduced in finance by Merton [29], or the more recent martingale methods, which were suggested as an alternative way to solve the optimal consumption/portfolio optimization problem. Martingale techniques were developed initially by Pliska [34], Cox and Huang [5] and Karatzas, Lehoczky and Shreve [22], and presented in great generality in Karatzas and Shreve [24]. See also Cvitanić and Zapatero [12] for a more elementary exposition.

In this paper we consider a very general first-best risk-sharing framework, with effort and project selection. The continuous-time papers with full information mentioned above are particular cases of our model. As in those papers, we take advantage of the similarity of the problem to the setting considered in Merton [29]: Optimal consumption in this paper would be equivalent to optimal effort, and optimal portfolio would correspond to project selection. In particular, we use the martingale methodology to solve the problem. This approach allows us to consider general utility functions for both agents and characterize the optimal contract. In principle, it can be applied to general semimartingale (and not just Markovian diffusion) settings. Martingale methods work particularly well in a complete markets setting. For our problem, that means the possibility to fully control the volatility of the aggregate wealth process ("project selection"). In addition, we allow an interplay between volatility selection and drift ("optimal effort") selection, and we allow separate, independent control of the drift. We also allow continuous consumption of the aggregate wealth process, and, similarly, we allow the contract to consist of a continuous salary payment. The generality of our model makes it suitable for many applications as particular cases, including optimal executive compensation and money management compensation.

One specific question we are interested in is when the optimal contract is non-linear. For the complete markets case we obtain detailed results, including an important example in which the individuals have different utility functions outside the class of CARA utilities, and the optimal contract is nonlinear and/or path dependent. Moreover, a wide range of contracts can be optimal, and when there is no independent drift control, it is optimal for the principal to offer a contract function that is a solution to a certain ordinary differential equation. This enables us to construct an example in which a call option contract is optimal.

A number of papers have extended martingale methods to incomplete markets setting (see section 5 for a review of some of these papers). In our setting, incomplete markets include the case in which there might be project selection subject to a cost function, but not choice of effort (like in Ou-Yang [32]), as well as the case in which there is effort, but not project selection (like in Holmström and Milgrom [20]). In order to address this type of problems we use an approach based on searching for a payoff which is related to a "fixed point" of a functional that connects the utility maximization problems of both individuals. It can also be represented as related to a solution of a corresponding dual problem. We re-derive the solution to the main example solved in Ou-Yang [32] using this approach and using ideas from Cvitanić [7] and Cuoco and Cvitanić [6]. We also address the original Holmström-Milgrom problem with exponential utilities and controlling the drift only. We illustrate our methodology by computing the principal's utility in the first-best case, which is strictly larger than the second-best, when the contracts are based solely on the observed controlled process.

The paper is organized as follows: In Section 2 we set up the model. In Section 3 we analyze the "complete markets " case. We then show that the first-best actions can be implemented by a contract of a simple form. We provide some explicit examples in Section 4. In Section 5 we analyze the general "incomplete markets" case. We conclude in Section 6, mentioning possible further research topics. The proofs of the main results are collected in the Appendix.

2 The Model

In this paper we consider first-best risk-sharing problems, which have many applications in practice. We do not consider agency issues, but it will be convenient (as in the other first-best problems in the literature reviewed in the previous section) to use the label "principal" for the individual who pays the compensation (such as the firm in an executive compensation problem or the investor in a money management problem) and the label "agent" for the other individual, in charge of exercising effort and selecting projects (such as the executive who works for the firm or the money manager who works for the investor).

We start with a very general model, but most of our results and all the examples will be for Brownian motion filtration. Let $\bar{X}$ be a $n-$dimensional RCLL semimartingale process on a probability space $(\Omega, \mathcal{F}, P)$, and denote by $\mathbf{F} := \{\mathcal{F}_t\}_{t \leq T}$ its augmented filtration on the interval $[0, T]$. Let us call the process controlled by the agent "stock price", motivated by the example of a company compensating its executives. The dynamics of the process $S = S^{a,\sigma,D}$ is given by

$$dS_t \;=\; \delta a_t dt - D_t dt + \theta(t, S_t)dt + \sigma_t' d\bar{X}_t \quad . \tag{1}$$

where $\delta \in [0, \infty)$ is a constant, and a, σ and D are $\mathbf{F}$-adapted stochastic processes chosen by the agent. Here, σ is $n-$dimensional and $\theta = \theta(t, S_t)$ is a possibly random functional of (t, S_t), such that the above equation has a unique solution for constant values of a, D and σ.

The process D represents the "dividend" rate or the "consumption" rate of the principal. The control a is the level of effort the agent applies. The higher a, the higher the expected value of the stock. We will assume later that the effort produces disutility for the agent. On the other hand, the choice of σ represents the choice of the volatility of the stock, although it may also have an impact on the expected value. We interpret the choice of σ as a choice of projects. We assume that the agent can choose different projects or strategies that are characterized by a level of risk and expected return. In the case of delegated portfolio management the value of δ and D would be zero. We study that case separately in a later section.

The agent receives final payment P_T from the principal, as well as a continuous payment at a "compensation" rate q. Here, P_T is a $\mathcal{F}_T-$measurable random variable, while q is an adapted process. We consider two optimization criteria for the agent: the "separable" criterion and the "non-separable" criterion. We are going to consider two problems. In Problem I, the agent wants to maximize, over a, σ, and D,

$$\textbf{Problem I:} \quad u_I(a,\sigma,D;P_T,q) := E\left[U_1\left(P_T\right) + \int_0^T [V_1(t,q_t) - \bar{G}(t,a_t,\sigma_t,S_t)]dt\right], \tag{2}$$

and in Problem II the agent wants to maximize over a, σ,

$$\textbf{Problem II:} \; u_{II}(a,\sigma,D;P_T,q) := u_{II}(a,\sigma,0;P_T,0) := E\left[U_1\left(P_T - \int_0^T \bar{G}(t,a_t,\sigma_t,S_t)dt\right)\right] . \tag{3}$$

Here, $U_1(\cdot)$ and $V_1(t,\cdot)$ are the utility functions of the agent, which we assume to be differentiable, strictly increasing and strictly concave. The function $\bar{G}$ is a penalty that measures the disutility from the effort a, but can also depend on t, σ_t, S_t. We assume that $\bar{G}(t,0,\sigma_t,S_t) = 0$ and $\bar{G}$ is a strictly convex and differentiable function of a, strictly increasing in $|a|$. We note that $D = q = 0$ in Problem II.

Problem I is a standard "separable" formulation, but to be able to include the Holmstrom-Milgrom [20] framework, we also consider the "non-separable" Problem II, which is common when exponential utilities are used.

In Problem I the principal's problem is to maximize over P_T and q

$$\textbf{Problem I:} \quad v_I(a,\sigma,D;P_T,q) \; := \; E\left[U_2(\beta S_T - P_T) + \int_0^T V_2(t,\kappa D_t - q_t)dt\right] \quad , \tag{4}$$

and in Problem II the principal's problem is to maximize over P_T

$$\textbf{Problem II:} \quad v_{II}(a,\sigma,D;P_T,q) \;:=\; v_{II}(a,\sigma,0;P_T,0) \;:=\; E\left[U_2(\beta S_T - P_T)\right]. \qquad (5)$$

Here, $U_2(\cdot)$ and $V_2(t,\cdot)$ are strictly increasing and strictly concave utility functions. In other words, the principal's utility measures the trade-off between the value of the stock at time T and the payoff P_T to the agent, as well as between the dividend rate D and the compensation rate q, where the relative importance of S_T and D is measured by constants $\beta > 0$, $\kappa > 0$. Here, $(1-\kappa)$ can also account for the tax rate on dividends.

The principal has full information and, in particular, she can observe the agent's actions. The time horizon T is fixed.

The principal has to guarantee that the (optimal) expected utility of the agent is at least as large as a reservation utility R. That is, for Problem I

$$\textbf{Problem I:} \quad \max_{a,\sigma,D} E\left[U_1\left(P_T\right) + \int_0^T [V_1(t,q_t) - \bar{G}(t,a_t,\sigma_t,S_t)]dt\right] \;\geq\; R,$$

and for Problem II

$$\textbf{Problem II:} \quad \max_{a,\sigma} E\left[U_1\left(P_T - \int_0^T \bar{G}(t,a_t,\sigma_t,S_t)dt\right)\right] \;\geq\; R.$$

The value R can be interpreted as the reservation utility that the agent would achieve in the best alternative offer he has. This restriction amounts to an *individual rationality constraint* or *participation constraint*. We will call it the “IR constraint”.

2.1 Relation to Merton's Problem

With full information, the principal can force the agent to implement the "first-best" solution, subject to the IR constraint. In other words, the principal solves the problem

$$\max_{a,\sigma,D,P_T,q}[v_i(a,\sigma,D;P_T,q)+\lambda u_i(a,\sigma,D;P_T,q)] \tag{6}$$

for $i = I$ or $i = II$, where the Lagrange multiplier λ will be chosen so that $u_i(a,\sigma,D;P_T,q) = R$.

Instead of finding optimal a,σ,D, we could consider finding the optimal stock price S_T and consumption process D which can be "financed" by some pair (a,σ). Suppose now that there is no effort a or that it is fixed, so that the only control is σ, and suppose $\bar{G} \equiv 0$. Consider, for concreteness, Problem I. Denote

$$U_\lambda(S) = \max_P[U_2(\beta S - P) + \lambda U_1(P)] \tag{7}$$

$$V_\lambda(t,D) = \max_q[V_2(t,\kappa D - q) + \lambda V_1(t,q)]. \tag{8}$$

Then, the first-best problem (for Problem I) is equivalent to the problem of maximizing

$$E\left[U_\lambda(S_T) + \int_0^T V_\lambda(t,D_t)dt\right] \tag{9}$$

where (S_T, D) can be financed by a "portfolio" σ in the sense that (1) is satisfied (with control a fixed). The problem (9) is of the form of the classical Merton [29] optimal portfolio and consumption selection problem, with utility functions U_λ and V_λ.

This interpretation provides useful intuition for the explicit results we obtain in examples. However, we do not use it explicitly in the proofs, which are provided for the general case with optimal effort a and with non-zero penalty function $\bar{G}$.

In order to solve the problem, we use "martingale" methods developed to address Merton's problem. Let us briefly recall the basic facts of the "martingale" approach. Suppose that in the model (1) the process a is fixed. It is well known in the finance literature (see, for example, Karatzas and Shreve [24]) that Merton's problem is much easier when markets are complete, since given D and a, any (sufficiently integrable) $\mathcal{F}_T$-measurable random variable S_T can be financed by a process σ, starting from a large enough initial amount S_0. It is also well known (as the "Second Fundamental Asset Pricing Theorem") that, under technical conditions, markets are complete (and arbitrage-free) in the model (1), with $\theta \equiv 0$, if and only if there exists a unique martingale process Z, so-called "Stochastic Discount Factor" or SDF, such that $Z_0 = 1$ and the process $M_t^S := Z_t S_t + \int_0^t Z_s[D_s - \delta a_s]ds$ is a martingale. In particular,

$$E[M_T^S] = E\left[Z_T S_T + \int_0^T Z_s(D_s - \delta a_s)ds\right] = S_0. \tag{10}$$

In such a model, the solution to the problem of maximizing (9) is to have

$$D_t = I_\lambda^V(t, yZ_t) \ , \qquad S_T = I_\lambda^U(yZ_T), \tag{11}$$

where $I_\lambda^U(\cdot)$, $I_\lambda^V(t,\cdot)$ are the inverse functions of marginal utilities $U_\lambda'(\cdot)$, $V_\lambda'(t,\cdot)$, and where the constant y is chosen so that (10) is satisfied.

On the other hand, if markets are incomplete, in the sense that we cannot attain all random variables S_T as the final value of process S (for an appropriate S_0), then there are many processes Z with above properties, and the one we need can typically be found as a solution to an associated dual problem.

In view of the above, we will first consider the complete markets case, that is, the case in which all (sufficiently integrable) outcomes S_T can be attained by appropriately choosing σ, if S_0 is such that (10) is satisfied. In those models, we will be able to solve more easily a number of examples, and to obtain more results than in the general case.

3 Complete Markets

In this section we assume that we have the following special case of model (1):

$$dS_t \ = \ \delta a_t dt - D_t dt + \alpha_t' \sigma_t dt + \sigma_t' dm_t \quad , \tag{12}$$

where m is a n-dimensional martingale process, and where α is a n-dimensional adapted process representing the size of the tradeoff between the risk and the return of S. For example, in one dimension higher α means higher return, but also higher risk. We could also allow a linear term of the form $\mu_t S_t dt$ in the dynamics for S, but this reduces to the dynamics of the above type by considering the "discounted" process $e^{-\int_0^t \mu_s ds} S_t$. In the examples below m will be a Brownian motion, possibly multi-dimensional. Another typical example would be a discrete-time model, with m being a random walk process.

We also assume that the penalty is only imposed on action (or effort) a:

$$\bar{G}(t, a, \sigma, S) = G(t, a).$$

Let us assume that for the model (12) there exists a martingale Z with $Z_0 = 1$ such that the process M defined by

$$M_t^S := Z_t S_t + \int_0^t Z_s [D_s - \delta a_s] ds \tag{13}$$

is a local martingale. We will impose a somehow stronger (technical) requirement:

Definition 3.1 *The set of admissible actions is the set of triplets* (a, σ, D) *such that process* M^S *is a martingale.*

Remark 3.1 As is well known, and can be checked by Itô's rule, if m is a one-dimensional

Brownian motion W, $m = W$, we can take

$$Z_t \;=\; \exp\left\{-\int_0^t \alpha_s dW_s - \frac{1}{2}\int_0^t \alpha_s^2 ds\right\}$$

satisfying

$$dZ_t \;=\; -\alpha_t Z_t dW_t \quad .$$

In this case we have

$$M_t^S \;=\; S_0 + \int_0^t (\sigma_s - \alpha S_s) Z_s dW_s, \tag{14}$$

and a sufficient condition for M^S to be a martingale is $E[\int_0^T |\sigma_s - \alpha S_s|^2 Z_s^2 ds] < \infty$. However, we will not have to check whether the candidate optimal triple is admissible, because the optimal solution will be constructed so that M^S is automatically a martingale.

We will also assume the following "spanning" property, or the martingale representation property (equivalent to complete markets):

Assumption 3.1 *Martingale Representation Property:* For any given pair (a, D) of adapted processes and any $\mathcal{F}_T$-measurable random variable X such that

$$E\left[Z_T X + \int_0^T Z_s(D_s - \delta a_s) ds\right] = S_0, \tag{15}$$

there is an adapted process σ such that $S_T^{a,D,\sigma} = X$ is the terminal value of the stock price process corresponding to (a, D, σ).

It is really this property which drives the results in this section, that is, *the ability of the agent to span, by an appropriate choice of* σ, all random outcomes X at time T for which the "budget constraint" (15) is satisfied.

In the case $m = W$, this property is satisfied as follows: Introduce a martingale

$$M_t^X \;:=\; E_t\left[Z_T X + \int_0^T Z_s(D_s - \delta a_s) ds\right] \tag{16}$$

where E_t denotes expectation conditional on the σ-algebra generated by W up to time t. The Martingale Representation Theorem says that we can write

$$M_t^X \;=\; E\left[M_T^X\right] + \int_0^t \varphi_s^X dW_s$$

for some adapted process φ^X. Having in mind (14), we choose σ so that

$$(\sigma_t - \alpha S_t)Z_t \;=\; \varphi_t^X \quad .$$

With such choice of σ, the corresponding stock process S will satisfy (1), we will have $M_t^X = M_t^S$ for all $t \leq T$, and $S_T = X$.

3.1 The First-Best solution

In order to describe the solution, denote by I_i^U the inverse function of the marginal utility function U_i', by $I_i^V(t,\cdot)$ the inverse function of the marginal utility function $V_i'(t,\cdot)$, and by $J(t,\cdot)$ the inverse function of $G'(t,\cdot)$, where G' denotes the derivative with respect to the second variable. That is,

$$I_i^U(z) := (U_i')^{-1}(z), \qquad I_i^V(t,z) := (V_i')^{-1}(t,z), \qquad J(t,x) := (G')^{-1}(t,x). \tag{17}$$

These inverse functions exist because U_i, $V_i(t,\cdot)$ are strictly concave and $G(t,\cdot)$ is strictly convex.

Introduce the indicator

$$\Delta_I = \begin{cases} 1 & \text{for Problem I} \\ 0 & \text{for Problem II} \end{cases} \tag{18}$$

and analogously for Δ_{II}. We will see below that the IR constraint becomes, for some constant $\hat{z}$,

$$R \;\leq\; E\left[U_1(I_1^U(\hat{z}Z_T)) + \Delta_I \times \int_0^T \left\{V_1\left(I_1^V\left(\frac{\beta}{\kappa}\hat{z}Z_s\right)\right) - G(s, J(s, \delta\beta\hat{z}Z_s))\right\} ds\right] \quad . \tag{19}$$

We need the following assumption to solve the principal's problem

Assumption 3.2 *There exists a unique number $\hat{z}$ such that* (19) *is satisfied as equality.*

We also impose the following assumption, whose meaning will become more clear in the proof of Theorem 3.1 in the Appendix, and which corresponds to (15):

Assumption 3.3 *There exists a number $y = \hat{y}$ so that the following principal's feasibility constraint is satisfied:*

$$\begin{aligned} &\beta[S_0 + \Delta_{II} \times T\delta J(T, \delta\beta)] + \Delta_I \times E\left[\int_0^T \beta\delta Z_t J\left(t, \beta\delta\hat{z}Z_t\right) dt\right] \\ = \;& E\left[Z_T\left\{I_1^U(\hat{z}Z_T) + I_2^U(\hat{y}Z_T) + \Delta_{II} \times TG(T, J(T, \delta\beta))\right\}\right] \\ &+\Delta_I \times E\left[\int_0^T \left\{Z_s\frac{\beta}{\kappa}\left[I_1^V\left(s, \frac{\beta}{\kappa}\hat{z}Z_s\right) + I_2^V\left(s, \frac{\beta}{\kappa}\hat{y}Z_s\right)\right]\right\} ds\right]. \end{aligned} \tag{20}$$

The main result in this subsection is

Theorem 3.1 *Suppose that Assumptions 3.2 and 3.3 hold. Then, the solution to the first-best problem* (6) *consists of the payoff*

$$\hat{P}_T \;=\; \beta S_T - I_2^U(\hat{y}Z_T) \quad , \tag{21}$$

and the compensation rate

$$\hat{q}_t \;=\; \Delta_I \times \left\{\kappa D_t - I_2^V\left(t, \frac{\beta}{\kappa}\hat{y}Z_t\right)\right\} \quad . \tag{22}$$

The optimal first-best triple $(\hat{a}, \hat{\sigma}, \hat{D})$ is given by

$$\kappa\hat{D}_t \;=\; \Delta_I \times \left\{I_1^V\left(t, \frac{\beta}{\kappa}\hat{z}Z_t\right) + I_2^V\left(t, \frac{\beta}{\kappa}\hat{y}Z_t\right)\right\} \quad , \tag{23}$$

$$\hat{a}_t \;\equiv\; \Delta_I \times J(t, \beta\delta\hat{z}Z_t) + \Delta_{II} \times J(t, \delta\beta) \quad , \tag{24}$$

while $\hat{\sigma}$ is chosen so that at the final time we have

$$\beta\hat{S}_T \;=\; I_1^U(\hat{z}Z_T) + I_2^U(\hat{y}Z_T) + \Delta_{II} \times TG(T, J(T, \delta\beta)) \quad . \tag{25}$$

Proof: See the Appendix. $\diamond$

Remark 3.2 (i) In all of our examples below Assumptions 3.2 and 3.3 are satisfied. For Problem II, Assumption 3.2 boils down to the existence of $\hat{z}$ such that $R = E[U_1(I_1^U(\hat{z}Z_T))]$, which is satisfied if R is in the range of function U_1, and the range of I_1^U covers the whole domain of U_1. Similarly, Assumption 3.3 is not hard to verify in case of Problem II. For example, it is sufficient in that case that the function $I_2^U(\cdot)$ has the whole real line as its range.

(ii) A standard agency problem is the moral hazard case in which the effort a is not observed by the principal. In our setting, that would imply that m is not observable either, and then our optimal contract $(\hat{P}, \hat{q})$ may not be feasible, since the process Z depends on m. However, notice that as long as there is no agency cost, our solution remains optimal, even if there is no effort a, since if S is continuously observable, then the control σ is observable, as a quadratic variation of process S. This is, for example, a natural setting for the analysis of the problem of delegated portfolio management.

(iii) In case of Problem I, from (3.10) and (3.14) it follows that

$$U_2'(\beta S_T - \hat{P}_T) = \frac{\hat{y}}{\hat{z}} U_1'(\hat{P}_T) \tag{26}$$

This is a familiar condition for Pareto-optimal risk-sharing between two agents.

3.2 Contracts That Implement the First-Best Solution

We say that a triple (a, σ, D) is *implementable* by a contract (P_T, q) if, when offered that contract, the agent chooses optimally the actions (a, σ, D). The issue of implementability is usually not considered in the full information literature because, with full information, the principal can force the agent to implement the first-best. However, it is of interest for practical applications to find simple contracts which implement the first-best. See Remark 3.3 below.

We now claim that the contract from Theorem 3.1 implements the first-best controls $(\hat{a}, \hat{\sigma}, \hat{D})$.

Proposition 3.1 *We suppose that Assumptions 3.2, 3.3 hold. If the principal offers the agent the contract* $(\hat{P}_T, \hat{q}_t)$ *of* (21) *and* (22)*, then the agent will choose the first-best controls* $(\hat{a}, \hat{\sigma}, \hat{D})$ *of Theorem 3.1.*

Proof: See the Appendix. $\diamond$

Remark 3.3 *(Uniqueness of the optimal contract.)* In our setting, there are many ways to achieve the first-best solution. For example, a contract that imposes an infinite penalty on the agent if the first-best control is not chosen. Here we want to consider only contracts that depend on the stock price and on some benchmark (independent of the effort). This is a class of contracts widely used in practice. They are of the type

$$P_T = F(S_T) - B_T, \tag{27}$$

and similarly for q, where $F(x)$ is a deterministic function, such that $F'(x) > 0$, and B_T is a $\mathcal{F}_T$ measurable random variable. That is, *the payoff depends on the performance of the underlying process* S *compared to a benchmark value* B_T. In Cvitanić, Wan and Zhang [10], which follows up on the present paper and extends it, it is shown, in the model with one Brownian Motion and with $q \equiv D \equiv 0$, that the contract (21) is the only contract of the type (27) which implements the first-best solution, when there is both effort choice and project selection. When there is only project choice, if $\beta = 1$, it is shown that all optimal contracts of the type (27) which implement the first-best solution have to be of the form

$$F(S_T) - B_T = kS_T + I_1^U(\hat{z}Z_T) - k[I_1^U(\hat{z}Z_T) + I_2^U(\hat{y}Z_T)] \tag{28}$$

for some positive number k.

Remark 3.4 *(Benchmark value.)* As indicated in the previous remark, the agent is rewarded based on the performance of S_T relative to the benchmark value $B_T = I_2^U(\hat{y}Z_T)$. Note that this benchmark value is of the form which is optimal for Merton's problem of maximizing utility from terminal wealth with utility function U_2. This, together with a similar observation for q, is what drives the agent to apply the controls which are first-best for the principal. The benchmark internalizes for the agent the cost of deviating from the first-best. Thus, the principal is able to get the utility from a random outcome of the same form, $I_2^U(\hat{y}Z_T)$, as if maximizing utility directly (without the agent), except that the value of $\hat{y}$ is adjusted to account for the agent's reservation utility. Also note that, except for the constant $\hat{y}$, the contract does not require that the principal knows the agent's utility, which is convenient in practice.

3.3 Payoff as a Function of S_T; The Case of Delegated Portfolio Management: $\delta = D = q = 0$

We discuss next when there is an optimal contract which depends only on the final value S_T.

Remark 3.5 Suppose we can solve equation (25) for a unique positive solution Z_T in terms of S_T, as some function

$$Z_T = h(S_T) \quad .$$

That will give us the form of the optimal contract in terms of S_T, using (21):

$$\hat{P}_T = f(S_T) := \beta S_T - I_2^U(\hat{y}h(S_T)) = I_1^U(\hat{z}h(S_T)) + \Delta_{II} \times TG(T, J(T, \delta\beta)) \quad . \tag{29}$$

Even though, in this case, the optimal payoff $\hat{P}_T$ turns out, ex post, to be of the form $P_T = f(S_T)$ for some function f, it is not true in general, that it is optimal to offer a contract to the agent in that form, ex ante. More precisely, if we want the contract to induce the agent to implement the first-best solution, the optimal risk-sharing rule does not involve a payment for the agent of the type $P_T = f(S_T)$, but of the type $\hat{P}_T = \beta S_T - I_2^U(\hat{y} Z_T)$. We show in the next section in a counterexample that the utility for the principal when $P_T = f(S_T)$ is offered can be smaller than the optimal utility resulting from the optimal contract of the previous theorem, even when f is linear, if $\delta > 0$. On the other hand, if $\delta = 0$, we will show that if $f(S_T)$ is a linear function, then $\hat{P}_T = f(S_T)$ is an optimal risk-sharing rule.

We assume now that we are still in a complete markets setting, but there is neither choice of effort a, nor dividend rate D, nor compensation rate q, and we call this model *the delegated portfolio management model*:

$$dS_t \;=\; \alpha_t' \sigma_t dt + \sigma_t' dm_t \quad . \tag{30}$$

Here, the process S should be interpreted as the value of a portfolio selected by a manager who uses the portfolio strategy σ. That is, the manager chooses the weights in the different securities available, which amounts to the choice of σ, the volatility of the portfolio, and also has an effect on the expected return of the portfolio. The process α represents the local expected return rate of S, a weighted average of the local expected returns of the securities in the portfolio. The process m represents the random shocks to the value of S.

We denote $I_i^U = I_i$. In the following, we say that a contract is *optimal for the principal* if the principal can attain the same maximal utility with that contract, as when the first-best solution is implemented. We have the following result about the optimal contract being a function of S_T:

Proposition 3.2 *Consider the delegated portfolio management model* (30)*. Assume that the agent is allowed to use only controls σ for which the local martingale process ZS is a martingale. Assume also that there exists a function $d = d(s)$ which satisfies the ordinary* *differential equation*

$$d'(s) \;=\; \frac{z_d}{\hat{y}}\frac{U_2'(\beta s - d(s))}{U_1'(d(s))}, \tag{31}$$

for a given constant z_d and $\hat{y}$ as before. In addition, assume that the maximum in

$$\bar{U}_1^d(z) \;=\; \max_s\{U_1(d(s)) - zs\} \tag{32}$$

is attained at a unique value $s = s(z)$ for which the first derivative of the function on the right-hand side is zero:

$$U_1'(d(s(z)))d'(s(z)) \;=\; z \quad . \tag{33}$$

Assume that z_d and the boundary condition for the solution $d(s)$ of the above ODE can be chosen so that

$$E[Z_T s(z_d Z_T)] \;=\; S_0$$

and

$$E[U_1(d(s(z_d Z_T)))] \;=\; R \quad . \tag{34}$$

Then, it is optimal for the principal to offer the contract

$$\hat{P}_T \;=\; d(S_T) \quad .$$

Proof: See the Appendix. $\diamond$

Remark 3.6 (i) It can be checked that if both agents have exponential utilities, or if they both have the same power utility, then the solution to the ODE (31) is a linear function, so that the optimal contract is then linear. Moreover, there are many cases in which the solution exists and is not linear. We postpone the details until we present examples in the next section. Numerical experiments suggest that, in general, there are no natural conditions under which the solution $d(s)$ is convex or concave (see Larsen [27] for some numerical results for this problem).

(ii) Note that (26) is different from (31) (except when $d(s)$ is linear). The ODE provides a contract which will result in the highest possible utility for the principal, but will not necessarily induce the agent to implement the first-best actions. Also note that the marginal payoff $d'(s)$ is proportional to the ratio of marginal utilities.

Next, we want to show, using a different argument, a related result: if the function f of (29) is linear, then the first-best risk-sharing rule can be determined as $f(S_T)$. First, we have the following corollary to Theorem 3.1.

Corollary 3.1 *Let assumptions of Theorem 3.1 hold and $\hat{y}$ and $\hat{z}$ be as in that theorem. In the delegated portfolio management model* (30)*, any contract for which at time T we have*

$$P_T \;=\; \beta S_T - I_2(\hat{y} Z_T) \tag{35}$$

is optimal for the principal. Reversely, if a contract is optimal for the principal, then it has to satisfy the above equality at time T.

Proof: See the Appendix. $\diamond$

Theorem 3.2 *Consider the delegated portfolio management model* (30)*. Assume that the agent is allowed to use only controls σ for which the local martingale process ZS is a martingale, and that there exists a number $\hat{y}$ such that the principal's feasibility constraint* (20) *is satisfied, with $\hat{z}$ determined from the IR constraint* (19) *satisfied as equality. Assume that the function $f(s) = \beta s - I_2(\hat{y} h(s))$ of* (29) *is a non-constant linear function, and that there is a unique solution z^* to the equation*

$$S_0 \;=\; E[Z_T f^{-1}(I_1(z^* Z_T))]. \tag{36}$$

Then, the linear contraxt $P_T = f(S_T)$ is optimal for the principal.

Proof: See the Appendix. $\diamond$

4 Examples

We only consider examples with $D \equiv q \equiv 0$, but they are easily extended to the general case.

Example 4.1 (*Power and Exponential Utilities.*) Consider first the case of power utilities in which, for $\gamma_1 < 1$, $\gamma_2 < 1$,

$$U_1(x) = \frac{1}{\gamma_1} x^{\gamma_1} \qquad \text{and} \qquad U_2(x) = \frac{1}{\gamma_2} x^{\gamma_2} \quad .$$

Then,

$$I_1(z) = z^{\frac{1}{\gamma_1 - 1}} \qquad \text{and} \qquad I_2(z) = z^{\frac{1}{\gamma_2 - 1}}.$$

Thus, (25) becomes

$$\beta S_T \;=\; (zZ_T)^{\frac{1}{\gamma_1 - 1}} + (yZ_T)^{\frac{1}{\gamma_2 - 1}} + \Delta_{II} \times TG(T, J(T, \delta\beta)) \quad .$$

In particular, if the utilities are the same, $\gamma_2 = \gamma_1$, we get

$$Z_T^{\frac{1}{\gamma_1 - 1}} \;=\; \frac{\beta S_T - \Delta_{II} \times TG(T, J(T, \delta\beta))}{z^{\frac{1}{\gamma_1 - 1}} + y^{\frac{1}{\gamma_1 - 1}}}$$

and

$$P_T \;=\; \beta S_T - I_2(yZ_T) \;=\; z^{\frac{1}{\gamma_1 - 1}} \frac{\beta S_T - \Delta_{II} \times TG(T, J(T, \delta\beta))}{z^{\frac{1}{\gamma_1 - 1}} + y^{\frac{1}{\gamma_1 - 1}}} + \Delta_{II} \times TG(T, J(T, \delta\beta)) \quad .$$

That is, if the principal and agent have the same utility, and they both behave optimally, the payoff turns out to be linear at time T. The payoff can be offered in this form if $\delta = 0$.[2] We note that if the principal and the agent have different power utilities, the solution will be a non-linear contract, ex post.

[2]Ross [35] shows this result in a static setting. He calls it "the principle of similarity."

One way to provide intuition for this is that, if the utility functions are the same, optimal P for the maximization in the definition (7) of function U_λ is a linear function of S. Thus, the optimal risk-sharing rule is linear. Similarly for the following example:

Consider the case of exponential utilities in which

$$U_1(x) = -\frac{1}{\gamma_1} e^{-\gamma_1 x} \qquad \text{and} \qquad U_2(x) = -\frac{1}{\gamma_2} e^{-\gamma_2 x} \quad .$$

Then,

$$I_1(z) = -\frac{1}{\gamma_1}\log(z) \qquad \text{and} \qquad I_2(z) = -\frac{1}{\gamma_2}\log(z).$$

Thus, (25) becomes

$$\beta S_T + \frac{1}{\gamma_1}\log(zZ_T) \;=\; -\frac{1}{\gamma_2}\log(yZ_T) + \Delta_{II} \times TG(T, J(T, \delta\beta)) \quad .$$

Solving for $\log(Z_T)$ in terms of S_T,

$$\log(Z_T) \;=\; \frac{\Delta_{II} \times TG(T, J(T, \delta\beta)) - \beta S_T - \frac{1}{\gamma_1}\log(z) - \frac{1}{\gamma_2}\log(y)}{\frac{1}{\gamma_1} + \frac{1}{\gamma_2}}.$$

Then, using $P_T = I_1(zZ_T) + \Delta_{II} \times \int_0^T G(s, \hat{a}_s)ds$, we get

$$\begin{aligned} P_T \;&=\; -\frac{1}{\gamma_1}\log(z) - \frac{1}{\gamma_1}\left\{\frac{\Delta_{II} \times TG(T, J(T, \delta\beta)) - \beta S_T - \frac{1}{\gamma_2}\log(y) - \frac{1}{\gamma_1}\log(z)}{\frac{1}{\gamma_2} + \frac{1}{\gamma_1}}\right\} \\ &\quad + \Delta_{II} \times TG(T, J(T, \delta\beta)) \quad . \end{aligned}$$

That is, if they have exponential utilities, when they both behave optimally the payoff turns out to be linear at time T, even when they have different risk aversions. This is consistent with the known risk-sharing results; it is also true in the case of hidden information, as shown in Sung [39]. (In his setting $\alpha = 0$. Here, this would mean that $Z_t \equiv 1$, so that the optimal first-best contract is $\beta S_T - I_2(\hat{y})$).

Example 4.2 (*A Counterexample for Linear Payoffs.*) We show here with an example the case discussed in Remark 3.5, where we argued that when $\delta > 0$ it may not be optimal to choose the contract $f(S_T)$ from (29), even when it is linear. Assume the model (12), but with $D \equiv 0$, $m = W$ being a one-dimensional Brownian Motion and α being a positive constant. Consider exponential utilities as in the previous example, and consider only Problem II for the agent, with $G(t, a) = G(a)$. Then the function f is linear,

$$f(S_T) \;=\; c + bS_T,$$

where c and b can be determined from that example:

$$c = \frac{\gamma_2^{-1}}{\gamma_2^{-1} + \gamma_1^{-1}} \left[TG(J(\delta\beta)) + \gamma_1^{-1} \log(\hat{y}/\hat{z})\right] \qquad \text{and} \qquad b = \beta \frac{\gamma_1^{-1}}{\gamma_2^{-1} + \gamma_1^{-1}}. \tag{37}$$

Let us suppose now that the contract offered is $f(S_T)$. We show in the Appendix that then the agent will choose optimal actions so that the principal's utility is of the form

$$E\left[U_2(\beta S_T - P_T)\right] \;=\; -\frac{\hat{y}}{\gamma_2} \exp\left\{-\gamma_1 \left[TG(J(\delta b)) - TG(J(\delta\beta)) + \frac{1}{\gamma_1} \log(\hat{z}/\tilde{z})\right]\right\} \tag{38}$$

and that this is strictly smaller than $-\frac{\hat{y}}{\gamma_2}$, which is the utility with the optimal contract $\hat{P}_T = \beta S_T - I_2(\hat{y} Z_T)$.

Example 4.3 (*Nonlinear Payoff as the ODE Solution.*) We show here by example that the ODE (31) can have a nonlinear solution, and hence that a payoff which is a nonlinear function of S_T can be optimal.

We assume that

$$U_1(x) = \log(x) \qquad \text{and} \qquad U_2(x) = -e^{-x} \quad .$$

Then the ODE (31) becomes

$$e^{-d(s)} \frac{d'(s)}{d(s)} \;=\; \frac{z_d}{\hat{y}} e^{-\beta s} \quad . \tag{39}$$

It can be seen from (20) that $\hat{y}$ has to be positive, and we assume that z_d is positive. Recall now a well known special function $Ei(x)$, called *exponential integral*, defined by

$$Ei(x) \ := \ -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt \quad . \tag{40}$$

This is a well defined function except at $x = 0$. We are only interested in $x < 0$, where this function is continuous and decreases from 0 to $-\infty$. In other words,

$$Ei(-\infty) = 0 \qquad \text{and} \qquad Ei(0) = -\infty \quad .$$

We note that, for every $x < 0$:

$$Ei'(x) = \frac{e^x}{x} \qquad \text{and} \qquad Ei''(x) = -\frac{e^x}{x^2} < 0.$$

We can see that

$$-Ei'(-x) \ = \ e^{-x}/x \quad ,$$

so that integrating (39) we get

$$Ei(-d(s)) \ = \ -\frac{z_d}{\hat{y}\beta} e^{-\beta s} + C \quad . \tag{41}$$

We take C to be a non-positive constant. Since U_1 is not defined on negative numbers, we want the potential contract $d(s)$ to be positive. Thus, we consider the inverse function Ei^{-1} on the domain $(-\infty, 0)$, with values in $(-\infty, 0)$. We see that

$$d(s) \ = \ -Ei^{-1}\left(-\frac{z_d}{\hat{y}\beta} e^{-\beta s} + C\right) \quad . \tag{42}$$

This is a well defined function on $s \in (-\infty, \infty)$, continuous and increasing, with

$$d(-\infty) = 0 \qquad \text{and} \qquad d(\infty) \in (0, \infty] \quad .$$

If $C < 0$ then $d(\infty) < \infty$, if $C = 0$ then $d(\infty) = \infty$. We verify the remaining assumptions of Proposition 3.2 in the Appendix, showing that the contract $d(S_T)$ is optimal.

Example 4.4 (*Call Option Contract.*) We still assume that $\delta = D = q = 0$. We can use Proposition 3.2 to ask ourselves a reverse question: For which utilities is a given contract $d(S_T)$ optimal? In this example we show that using log utilities an option-like contract

$$d(s) = n(s - K)^+$$

can be optimal, for some $K > 0, n > 0$. We assume

$$S_0 > K \qquad \text{and} \qquad \beta > n \quad .$$

Suppose that

$$U_1(x) = c \log(x)$$

for some $c > 0$. Then,

$$I_1(z) = c/z,$$

and we can see that the maximum in

$$\bar{U}_1^d(z) \;=\; \max_{s>0} \{U_1(d(s)) - sz\}$$

is $\hat{s} = c/z + K$, which means, as in proof of Proposition 3.1 in the Appendix, that the agent will act so that

$$S_T \;=\; \frac{c}{z_d Z_T} + K \tag{43}$$

for the value of z_d for which $E[Z_T S_T] = S_0$, which is equivalent to

$$\frac{c}{z_d} \;=\; S_0 - K \quad . \tag{44}$$

Analyzing the ODE from Proposition 3.2, we see that we should try the utility of the form $U_2(x) = b\log(x - c_1)$ for the principal. More precisely, let us assume that

$$U_2(x) = \begin{cases} b\log(x - \beta K) & \text{if } x > \beta K \\ -\infty & \text{if } x \le \beta K. \end{cases} \tag{45}$$

Here, $b > 0$ is a constant. Note that

$$I_2(z) \;=\; \frac{b}{z} + \beta K \quad .$$

Now, the principal gets his utility from

$$\beta S_T - n(S(T) - K)^+ \;=\; (\beta - n)\frac{c}{z_d Z_T} + \beta K \quad ,$$

by (43), so we would like this expression to be equal to $I_2(\hat{y} Z_T)$, since this gives the maximum utility for the principal. This will be true if

$$\frac{(\beta - n)c}{z_d} = \frac{b}{\hat{y}}. \tag{46}$$

Here, $\hat{y}$ has to satisfy the original principal's feasibility constraint (see (20))

$$\beta S_0 \;=\; E[Z_T I_1(\hat{z} Z_T) + Z_T I_2(\hat{y} Z_T)], \tag{47}$$

where $\hat{z}$ satisfies the original IR constraint (see (19))

$$E\left[c\log\left(\frac{c}{\hat{z} Z_T}\right)\right] \;=\; R \quad . \tag{48}$$

On the other hand, the IR constraint with the present contract is

$$E\left[U_1(n(S_T - K)^+)\right] \;=\; E\left[c\log\left(\frac{nc}{z_d Z_T}\right)\right] \;=\; R \quad .$$

From the last two equations we see that we need to have

$$n\hat{z} = z_d \quad ,$$

and if one of the equations is satisfied, the other will be, too. With $\hat{z} = z_d/n$, using (46), we see that the condition (47) becomes

$$\beta S_0 \; = \; \beta K + \beta \frac{c}{z_d} \quad ,$$

which is satisfied because of (44). Finally, using this, we see that condition (48) becomes

$$R \; = \; E\left[c\log\left(\frac{n(S_0 - K)}{Z_T}\right)\right] . \tag{49}$$

For example, in case $m = W$, this reads

$$R \; = \; c\left[\log\{n(S_0 - K)\} + \alpha^2 T/2\right] \quad . \tag{50}$$

To recap, this is what we have shown: Assuming $\delta = D = q = 0$, consider any values $c > 0$, $K < S_0$, $\beta > n$ such that (49) is satisfied. Then, if the agent and the principal have the utilities $U_1(x) = c\log(x)$, $U_2(x) = b\log(x - \beta K)$ respectively, for some arbitrary constant $b > 0$, then the option-like contract $d(s) = n(s - K)^+$ is optimal. Here, the value of b is arbitrary because only the ratio $b/\hat{y}$ matters, as seen from (46), and from the fact that (47) is equivalent to (44) and thus does not identify $\hat{y}$ separately.

From (45) we can interpret βK as the lower bound on his wealth that the principal is willing to tolerate. We see that the higher this bound, the higher the strike price K will be.

Remark 4.1 *(Optimal contract cannot be linear.)* One might wonder if it is possible to attain the best possible utility for the principal by offering a linear contract $c_1 S_T + c_2$, by trying to solve for c_1, c_2 from the IR constraint and from matching the optimal utility value. We can use the previous example to show that this is typically not possible. With

$c = b = n = 1, \beta = 2$, and $m = W$, it can be computed that the principal's optimal utility value in that example is $\log(S_0 - K) + \alpha^2 T/2$, while the utility of the linear contract (satisfying the IR constraint), if the agent behaves optimally, is

$$E\left[\log\left\{(S_0 - K)\left(\frac{2}{c_1} - 1\right)\left(\frac{1}{Z_T} - 1\right) + S_0 - K\right\}\right].$$

If $\alpha \neq 0$, $1/Z_T - 1$ can attain any value, and the utility would be $-\infty$, unless $c_1 = 2$. The maximal utility is thus $\log(S_0 - K)$ which is less than the optimal utility.

5 General Models

In this section we consider the general model (1) (with incomplete markets, possibly), and suggest a duality approach for solving the first-best risk-sharing problem in that context. We also present an example in which the problem solved in Ou-Yang [32] is solved here using our approach. His approach uses Hamilton-Jacobi-Bellman partial differential equations, and is thus restricted to diffusion models, while our approach can be applied to general models, even though it may not lead to explicit solutions.

In Problem I, we again assume that the penalty is only imposed on action a, $\bar{G}(t, a, \sigma, S) = G(t, a)$. In Problem II, we consider the general cost function $\bar{G}(t, a, \sigma, S)$.

In order to describe a candidate solution, we introduce some notation. Let us denote by $\mathcal{D}_L$ the set of all adapted stochastic processes Z for which, using notation (18),

$$E\left[Z_T\left(\beta S_T - \Delta_{II} \times \int_0^T \bar{G}(t, a_t, \sigma_t, S_t)dt\right) + \Delta_I \times \beta \int_0^T Z_t(D_t - \delta a_t)dt\right] \leq L(Z) \quad (51)$$

for some real valued functional $L(\cdot)$ acting on the process Z, such that $L(Z)$ is independent of a, σ, D, S, and such that

$$\forall \eta \in (0, \infty): \qquad L(\eta Z) = \eta L(Z). \quad (52)$$

(We are denoting by Z the stochastic process, and by Z_t the random variable which is the value of the stochastic process Z at time t.) For example, if $a \equiv \bar{G} \equiv D \equiv 0$, condition (51) boils down to $E[Z_T S_T] \leq L(Z)$.

For a given process $Z \in \mathcal{D}_L$, consider the optimization problem

$$H(Z) := \qquad (53)$$
$$\inf_{Y \in \mathcal{D}_L} E\left[L(Y) + \tilde{U}_1(Y_T) - Y_T I_2^U(Z_T) + \Delta_I \times \int_0^T \left\{\tilde{V}_1\left(s, \frac{\beta}{\kappa} Y_s\right) - Y_s I_2^V\left(\frac{\beta}{\kappa} Z_s\right) + \tilde{G}(s, Y_s)\right\} ds\right].$$

Here, we have introduced dual functions

$$\tilde{U}_1(z) := \max_x \{U_1(x) - xz\}, \quad \tilde{V}_1(t,z) := \max_y \{V_1(t,y) - yz\}, \quad \tilde{G}(t,z) := \max_a \{\beta\delta a z - G(t,a)\}. \qquad (54)$$

Consider now those stochastic processes Z for which the infimum in (53) is attained uniquely at some stochastic process $\hat{Y}(Z) =: F(Z)$, where we just defined a new functional F. We have the following main result of this section.

Theorem 5.1 *Assume that in the case of Problem I we have $\bar{G}(t, a, \sigma, S) = G(t, a)$. Fix a functional L as above. Suppose that the mapping $F(\cdot)$ has a "fixed point" $\hat{Z}$ in the sense that*

$$F(\hat{Z}) = \bar{c}\hat{Z} \qquad (55)$$

for some positive constant $\bar{c}$. Let the dividend rate be given by

$$\kappa \hat{D}_t = \Delta_I \times \left\{ I_1^V\left(t, \frac{\beta}{\kappa}\bar{c}\hat{Z}_t\right) + I_2^V\left(t, \frac{\beta}{\kappa}\hat{Z}_t\right)\right\} , \qquad (56)$$

and in case of Problem I, let the effort be given by

$$\hat{a}_t = J\left(t, \beta\delta\bar{c}Z_t\right) , \qquad (57)$$

where $J(t,\cdot)$ is the inverse function of $\frac{\partial}{\partial a}G(t,\cdot)$. Suppose also that, for Problem I there exists *an admissible process $\hat{\sigma}$, and for Problem II there exist admissible processes $\hat{a}, \hat{\sigma}$, such that*

$$\beta S_T \; = \; I_1(\bar{c}\hat{Z}_T) + I_2(\hat{Z}_T) + \Delta_{II} \times \int_0^T \bar{G}(t, \hat{a}_t, \hat{\sigma}_t, S_t)dt \tag{58}$$

and (51) *is satisfied as equality:*

$$E\left[\hat{Z}_T\left(\beta S_T - \Delta_{II} \times \int_0^T \bar{G}(t, \hat{a}_t, \hat{\sigma}_t, S_t)dt\right) + \Delta_I \times \beta \int_0^T \hat{Z}_t(\hat{D}_t - \delta \hat{a}_t)dt\right] \; = \; L(\hat{Z}) \tag{59}$$

Suppose also that the IR constraint is satisfied as equality:

$$R \; = \; E\left[U_1(I_1^U(\bar{c}\hat{Z}_T)) + \Delta_I \times \int_0^T \left\{V_1\left(I_1^V(\frac{\beta}{\kappa}\bar{c}\hat{Z}_t)\right) - \bar{G}(t, \hat{a}_t, \hat{\sigma}_t, S_t)\right\} dt\right] \quad . \tag{60}$$

Then, the first-best solution is the multiple $(\hat{a}, \hat{\sigma}, \hat{D})$ and the contract

$$\hat{P}_T \; = \; \beta S_T - I_2^U(\hat{Z}_T) \;, \quad \hat{q}_t \; = \; \Delta_I \times \left\{\kappa \hat{D}_t - I_2^V\left(t, \frac{\beta}{\kappa}\hat{Z}_t\right)\right\} \tag{61}$$

implements the first-best solution. Moreover, the pair $(\bar{c}^{-1}, \hat{Z})$ is the solution to the dual problem

$$\inf_{\lambda > 0, Z \in \mathcal{D}_L} \left\{L(Z) + E\left[U_2^*(Z_T, \lambda) + \Delta_I \times \int_0^T \{V_2^*(t, Z_t, \lambda) + \tilde{G}(t, Z_t)\}dt\right] - \lambda R\right\} \tag{62}$$

where

$$U_2^*(y, \lambda) \; := \; \max_{s,p}\{U_2(\beta s - p) + \lambda U_1(p) - y\beta s\} \tag{63}$$

$$V_2^*(t, \bar{y}, \lambda) \; := \; \max_{D,q}\{V_2(t, \kappa D - q) + \lambda V_1(t, q) - \bar{y}\kappa D\}\,. \tag{64}$$

Proof: See the Appendix. $\diamond$

Remark 5.1 The existence of the optimal $\hat{Y}(Z)$ and $\hat{Z}$ for the problems (53) and (62), and the existence of the corresponding $\hat{a}, \hat{\sigma}$, has been studied in various papers in different models, in the context of a dual problem to Merton's problem of maximizing expected utility for an investor trading in financial markets, initially in Brownian motion models, and recently in more general semimartingale models. Brownian motion models where the number of Brownian motion processes driving the model is higher than the dimension of the control vector σ were analyzed in He and Pearson [18] and Karatzas et al [23]. The case in which the control vector σ is constrained to take values in a convex set was resolved in Cvitanić and Karatzas [12]. See also the book Karatzas and Shreve [24]. In those papers there is no cost in applying σ. The case corresponding to the cost function $\bar{G} = \bar{G}(\sigma, S)$ being nonlinear was studied in Cvitanić [7] and Cuoco and Cvitanić [6]. Models more general than Brownian motion filtration have been studied in Kramkov and Schachermayer [26], Cvitanić, Schachermayer and Wang [9], Hugonnier and Kramkov [21], Karatzas and Žitković [25], among others. See also Duffie, Geoffard and Skiadas [15] and Dumas, Uppal and Wang [16] for recursive preferences. Let us also mention that in Cvitanić, Wan and Zhang [10] this existence problem is shown to be equivalent to the existence of a system of Forward-Backward Stochastic Differential Equations. These are typically very difficult problems, so that we expect that general results on the existence of the fixed point in the above theorem will be difficult to obtain. This is left for future research, while here we present examples in which the fixed point exists, and the approach works.

5.1 Examples

5.1.1 Example: Exponential Utility with volatility and size penalty

We illustrate here the power of the above approach by considering an example which was solved in Ou-Yang [32] using Hamilton-Jacobi-Bellman partial differential equations. For ease of notation we assume that $\beta = 1$. The underlying process is driven by a d-dimensional Brownian Motion W:

$$dS_t = rS_t dt + \alpha' \sigma_t dt + \sigma_t' dW_t \quad ,$$

where $a'b$ denotes the inner product of two d-dimensional vectors a and b. We also consider Problem II with a penalty of the type

$$\bar{G}(t,\sigma,s) \;=\; g(t,\sigma)+\gamma s \quad ,$$

that is, linear in the size S of the controlled process.

We will show that the contract (72) below is optimal. We want to find a candidate process $\hat{Z}$ corresponding to the optimal contract. Motivated by the results of Cvitanić [7] and Cuoco and Cvitanić [6], we consider an adapted vector processes $\lambda = \{\lambda_t; t \geq 0\}$ such that

$$E\left[\int_0^T \|\lambda_s\|^2 ds\right] < \infty \quad ,$$

and define the process Z^λ by

$$Z_t^\lambda \;:=\; \exp\left\{-\int_0^t (\alpha+\lambda_s)' dW_s - \frac{1}{2}\int_0^t \|\alpha+\lambda_s\|^2 ds\right\} \quad ,$$

satisfying

$$dZ_t^\lambda = -Z_t^\lambda(\alpha+\lambda_t)' dW_t \quad , \quad Z_0^\lambda = 1 \quad .$$

Next, we want to get upper an bound of the form (51). Integration by parts implies

$$\begin{aligned}\int_0^T \gamma S_t dt \;&=\; \int_0^T \gamma e^{rt}(e^{-rt}S_t)dt \\ &=\; \frac{\gamma}{r}(e^{rT}-1)e^{-rT}S_T - \int_0^T \frac{\gamma}{r}(1-e^{-rt})[\alpha'\sigma_t dt + \sigma_t' dW_t] \quad .\end{aligned}$$

Moreover, using this and Itô's rule, we can see that

$$\begin{aligned}d\left(Z_t^\lambda \int_0^t [\gamma S_s + g(s,\sigma_s)]ds\right) \;&=\; d\left(Z_t^\lambda \frac{\gamma}{r}(e^{rt}-1)e^{-rt}S_t\right) + Z_t^\lambda\left[\frac{\gamma}{r}(1-e^{-rt})\lambda_t'\sigma_t + g(t,\sigma_t)\right]dt \\ &\quad + (\ldots)dW_t \quad .\end{aligned}$$

Also, by Itô's rule,

$$d\left(Z_t^\lambda e^{-rt}S_t\right) \;=\; -Z_t^\lambda e^{-rt}\lambda_t'\sigma_t dt + (\ldots)dW_t \quad .$$

Using the last two equations, and assuming that all $(\ldots)dW_t$ terms are martingales (not just local martingales), so that their expectation is zero, we get

$$\begin{aligned}
&E\left[Z_T^\lambda\left(S_T - \int_0^T(\gamma S_t + g(t,\sigma_t))dt\right)\right] \\
&= \left(e^{rT} - \frac{\gamma}{r}(e^{rT}-1)\right)E[Z_T^\lambda e^{-rT}S_T] - E\left[\int_0^T Z_t^\lambda\left[\frac{\gamma}{r}(1-e^{-rt})\lambda_t'\sigma_t + g(t,\sigma_t)\right]dt\right] \\
&= \left(e^{rT} - \frac{\gamma}{r}(e^{rT}-1)\right)S_0 - E\left[\int_0^T Z_t^\lambda\left[\left(\frac{\gamma}{r} + (1-\frac{\gamma}{r})e^{r(T-t)}\right)\lambda_t'\sigma_t + g(t,\sigma_t)\right]dt\right] \quad .
\end{aligned} \tag{65}$$

Denote

$$f(t) \;=\; \frac{\gamma}{r} + (1-\frac{\gamma}{r})e^{r(T-t)} \quad .$$

As in Cvitanić [7] and Cuoco and Cvitanić [6], we define the dual function

$$\tilde{g}(t,\lambda) \;:=\; \max_\sigma\{-g(t,\sigma) - f(t)\sigma'\lambda\}$$

for those vectors λ for which this is well defined (not equal to infinity), which then make up the effective domain of $\tilde{g}$. We now consider only those vector processes $\lambda = \{\lambda_t; t \geq 0\}$ which take values in the effective domain of $\tilde{g}$. By the definition of the dual function and (65), we get

$$\begin{aligned}
&E\left[Z_T^\lambda\left(S_T - \int_0^T(\gamma S_t + g(t,\sigma_t))dt\right)\right] \\
&\leq\; L(Z^\lambda) := \left(e^{rT} - \frac{\gamma}{r}(e^{rT}-1)\right)S_0 + E\left[\int_0^T Z_t^\lambda \tilde{g}(t,\lambda_t)dt\right] \quad .
\end{aligned} \tag{66}$$

This is an upper bound of the form (51).

Hence, for a given stochastic process ρ and given constants z, y, we want to do the following minimization:

$$H(Z^\rho) \;:=\; \inf_\lambda E\left[\tilde{U}_1(zZ_T^\lambda) + L(zZ^\lambda) - zZ_T^\lambda I_2(yZ_T^\rho)\right] \tag{67}$$

as in (53). Assume now that the utilities are exponential and the cost function is quadratic:

$$U_i(x) = -\frac{1}{\gamma_i}e^{-\gamma_i x} \qquad \text{and} \qquad g(t,\sigma) = \frac{1}{2}\sigma' k_t \sigma \quad ,$$

for some matrix-valued function k_t. Note that the solution for the problem of minimizing a quadratic form

$$y'x + \frac{1}{2}x'kx$$

over x is given by

$$\hat{x} \;=\; -\left(\frac{k+k'}{2}\right)^{-1} y \quad . \tag{68}$$

Thus the maximum in the definition of $\tilde{g}$ is attained for, suppressing dependence on t,

$$\hat{\sigma} \;=\; -f\left(\frac{k+k'}{2}\right)^{-1} \lambda \quad , \tag{69}$$

and we have

$$\tilde{g}(t,\lambda) \;=\; f^2\lambda'\left[\left(\frac{k+k'}{2}\right)^{-1} - \left(\frac{k+k'}{2}\right)^{-1}\frac{k}{2}\left(\frac{k+k'}{2}\right)^{-1}\right]\lambda \quad .$$

In our case

$$\tilde{U}_i(z) = -\frac{1}{\gamma_i}z + \frac{1}{\gamma_i}z\log(z) \qquad \text{and} \qquad I_i(z) = -\frac{1}{\gamma_i}\log(z) \quad .$$

Thus, assuming again that all local martingales are martingales, the dual problem (67) is equivalent to minimizing

$$E\left[\frac{1}{\gamma_1} z Z_T^\lambda \log(z Z_T^\lambda) + \frac{1}{\gamma_2} z Z_T^\lambda \log(y Z_T^\rho) + \int_0^T z Z_t^\lambda \tilde{g}(t,\lambda_t)dt\right] \quad .$$

This, in turn, is equivalent to minimizing

$$\begin{aligned} & E\left[\int_0^T Z_t^\lambda \left(\frac{1}{2\gamma_1}\|\alpha+\lambda_s\|^2 - \frac{1}{\gamma_2}\left[\|\alpha+\rho_s\|^2/2 - (\alpha+\lambda_s)'(\alpha+\rho_s)\right]\right) ds\right] \\ + \quad & E\left[\int_0^T Z_t^\lambda \left(f_s^2 \lambda_s' \left[\left(\frac{k_s+k_s'}{2}\right)^{-1} - \left(\frac{k_s+k_s'}{2}\right)^{-1} \frac{k_s}{2}\left(\frac{k_s+k_s'}{2}\right)^{-1}\right]\lambda_s\right) ds\right] \quad . \end{aligned}$$

We conjecture now that the optimal λ is deterministic, if ρ is deterministic. That means we simply have to maximize the quadratic form in the integral above, and by (68), the optimal λ is given by (suppressing dependence on t)

$$-\lambda \;=\; \left[\frac{1}{\gamma_1}\mathbf{i} + f^2\left(\frac{k+k'}{2}\right)^{-1}\right]^{-1}\left(\frac{\alpha}{\gamma_1} + \frac{1}{\gamma_2}(\alpha+\rho)\right) \quad ,$$

where $\mathbf{i}$ is the identity matrix. It can now be verified that this is indeed the optimal $\lambda = \lambda(\rho)$ for a given deterministic process ρ, for example by checking that the Hamilton-Jacobi-Bellman equation is satisfied.

The fixed point is obtained by setting $\rho = \lambda$, which gives

$$\hat{\lambda} \;=\; -\Gamma\left[\Gamma\mathbf{i} + f^2\left(\frac{k+k'}{2}\right)^{-1}\right]^{-1}\alpha \tag{70}$$

where

$$\Gamma \;:=\; \frac{\gamma_1+\gamma_2}{\gamma_1\gamma_2} \quad . \tag{71}$$

We would like to use Theorem 5.1 to claim that the contract

$$\hat{P}_T \;=\; S_T - I_2(\hat{y} Z_T^{\hat{\lambda}}) \;=\; S_T + \frac{1}{\gamma_2}\left[\log(\hat{y}) - \int_0^T \|\alpha+\hat{\lambda}_s\|^2/2ds - \int_0^T (\alpha+\hat{\lambda}_s)'dW_s\right] \tag{72}$$

is optimal for an appropriate choice of $\hat{y}$, to be determined below. It can be verified that this is the same contract as in Ou-Yang [32], except that the term dW_t there is expressed in terms of a different process, having the interpretation of stock prices. Also, in his framework, the process S has the interpretation of the value of a managed fund.

We would like to check that the conditions (58) and (59) are satisfied with the stochastic processes Z and Y defined by $Z_t = \hat{z} Z_t^{\hat{\lambda}}$ and $Y_t = \hat{y} Z_t^{\hat{\lambda}}$, for some values of $\hat{z}, \hat{y}$. We choose $\hat{z}$ so that the IR constraint (60) is satisfied. In order to satisfy (58) we need to have

$$S_T \;=\; I_1(\hat{z} Z_T^{\hat{\lambda}}) + I_2(\hat{y} Z_T^{\hat{\lambda}}) + \int_0^T [g(s,\sigma_s) + \gamma S_s] ds \quad ,$$

or, after integration by parts,

$$\begin{aligned}\left(e^{rT} - \frac{\gamma}{r}(e^{rT}-1)\right) e^{-rT} S_T \;&=\; I_1(\hat{z} Z_T^{\hat{\lambda}}) + I_2(\hat{y} Z_T^{\hat{\lambda}}) \\ &+ \int_0^T \left[g(s,\sigma_s) - \frac{\gamma}{r}(1-e^{-rs})\alpha'\sigma_s'\right] ds - \int_0^T \frac{\gamma}{r}(1-e^{-rs})\sigma_s' dW_s \quad .\end{aligned}$$

Using

$$e^{-rT} S_T \;=\; S_0 + \int_0^T e^{-rs}\alpha'\sigma_s ds + \int_0^T e^{-rs}\sigma_s' dW_s$$

and substituting for I_1 and I_2, we need to have

$$\begin{aligned}&\left(e^{rT} - \frac{\gamma}{r}(e^{rT}-1)\right)\left(S_0 + \int_0^T e^{-rs}\alpha'\sigma_s ds + \int_0^T e^{-rs}\sigma_s' dW_s\right) \\ &= \; -\frac{1}{\gamma_1}\log(\hat{z}) - \frac{1}{\gamma_2}\log(\hat{y}) + \frac{\Gamma}{2}\int_0^T \|\alpha + \hat{\lambda}_s\|^2 ds \\ &\quad +\Gamma\int_0^T (\alpha+\lambda_s)' dW_s + \int_0^T \left[g(s,\sigma_s) - \frac{\gamma}{r}(1-e^{-rs})\alpha'\sigma_s'\right] ds - \int_0^T \frac{\gamma}{r}(1-e^{-rs})\sigma_s' dW_s \quad .\end{aligned} \tag{73}$$

By (69), we conjecture that we have to take σ to be

$$\hat{\sigma} \;=\; -f\left(\frac{k+k'}{2}\right)^{-1}\hat{\lambda} \;=\; f\left[\frac{f^2}{\Gamma}\mathbf{i} + \frac{k+k'}{2}\right]^{-1}\alpha \quad .$$

Indeed, it can now be verified, using the fact that

$$(\alpha + \hat{\lambda})\Gamma \;=\; f\hat{\sigma} \quad ,$$

that if we choose this value for σ, then in the equation (73) the dW integrals cancel out. In order for the remaining terms to satisfy equation (73), we need to have

$$\begin{aligned} 0 \;=\; & \left(e^{rT} - \frac{\gamma}{r}(e^{rT} - 1)\right) S_0 + \int_0^T \left[f(s)\alpha'\hat{\sigma}_s - \frac{1}{2\Gamma}\int_0^T f^2(s)\|\hat{\sigma}_s\|^2 - \frac{1}{2}\hat{\sigma}_s' k_s \hat{\sigma}_s \right] ds \\ & + \frac{1}{\gamma_1}\log(\hat{z}) + \frac{1}{\gamma_2}\log(\hat{y}) \quad . \end{aligned}$$

This is an equation in $\hat{y}$ that can be solved. It is now easily verified that (59) is also satisfied. Thus, by Theorem 5.1 the contract (72) is indeed optimal.

5.1.2 Example: Original Holmstrom-Milgrom framework

In this example we show that if we restrict the principal to observe only the controlled process S, but not the driving Brownian motion W, then his utility may be strictly lower than if he can offer contracts based on both S and W. We consider a one-dimensional case of the Holmstrom and Milgrom [20] setting, where only the drift is controlled,

$$dS_t \;=\; a_t dt + dW_t$$

and W is a one-dimensional Brownian motion. Note that this is the case of extreme “incompleteness”, in the sense that the volatility is completely fixed, not controlled.

The computations are similar, but simpler than in the previous example, and we omit the details. We could also solve the multi-dimensional case, as in that example. We set $\beta = 1$ for notational simplicity. We denote similarly as in the previous example,

$$Z_t^{\lambda} \;=\; \exp\left\{ -\int_0^t \lambda_s dW_s - \frac{1}{2}\int_0^t \lambda_s^2 ds \right\} \quad .$$

We consider Problem II with cost function $\bar{G}$ defined by $\bar{G}(a) = a^2/2$. According to Itô's rule,

$$d\left(Z_t^\lambda\left(S_t - \int_0^t \frac{1}{2}a_s^2 ds\right)\right) \;=\; Z_t^\lambda\left[a_t - a_t^2/2 - \lambda_t\right]dt + (\ldots)dW_t \quad .$$

Using this and the fact that $a - a^2/2$ is maximized at $\hat{a} = 1$, and assuming that all $(\ldots)dW_t$ terms are martingales (not just local martingales), so that their expectation is zero, we get

$$E\left[Z_T^\lambda\left(S_T - \int_0^T \frac{1}{2}a_s^2 ds\right)\right] \;\leq\; \frac{T}{2} + S_0 - E\left[\int_0^T Z_t^\lambda \lambda_t dt\right] \quad .$$

This is an upper bound of the form (51). Hence, for a given stochastic process ρ and given constants z, y, we want to do the following minimization:

$$G(Z^\rho) \;:=\; \inf_\lambda E\left[\tilde{U}_1(zZ_T^\lambda) - z\int_0^T Z_t^\lambda \lambda_t dt - zZ_T^\lambda I_2(yZ_T^\rho)\right] \tag{74}$$

similarly as in (53). Assume now that the utilities are exponential,

$$U_i(x) \;=\; -\frac{1}{\gamma_i}e^{-\gamma_i x} \quad .$$

We have then

$$\tilde{U}_i(z) = -\frac{1}{\gamma_i}z + \frac{1}{\gamma_i}z\log(z) \qquad \text{and} \qquad I_i(z) = -\frac{1}{\gamma_i}\log(z) \quad .$$

Thus, assuming again that all local martingales are martingales, the dual problem (74) is equivalent to minimizing

$$E\left[\frac{1}{\gamma_1}zZ_T^\lambda\log(zZ_T^\lambda) + \frac{1}{\gamma_2}zZ_T^\lambda\log(yZ_T^\rho) - z\int_0^T Z_t^\lambda \lambda_t dt\right] \quad .$$

This, in turn, is equivalent to minimizing

$$E\left[\int_0^T Z_t^\lambda \left(\frac{1}{2\gamma_1}\lambda_s^2 - \frac{1}{\gamma_2}\left[\frac{\rho_s^2}{2} - \lambda_s\rho_s\right]\right) ds\right] - E\left[\int_0^T Z_t^\lambda \lambda_t dt\right] \quad .$$

We again conjecture that the optimal λ is deterministic, if ρ is deterministic. That means that we simply have to maximize the quadratic function in the integral above, and the optimal λ is given from (suppressing dependence on t)

$$\frac{\lambda}{\gamma_1} + \frac{\rho}{\gamma_2} \;=\; 1 \quad .$$

It can now be verified that this is indeed the optimal $\lambda = \lambda(\rho)$ for a given deterministic process ρ, for example by checking that the Hamilton-Jacobi-Bellman equation is satisfied.

The fixed point is obtained by setting $\rho = \lambda$, which gives

$$\hat{\lambda} \;=\; \frac{1}{\Gamma} \tag{75}$$

where

$$\Gamma \;:=\; \frac{\gamma_1 + \gamma_2}{\gamma_1\gamma_2} \quad . \tag{76}$$

By Theorem 5.1, the contract

$$\hat{P}_T \;=\; S_T - I_2(\hat{y}Z_T^{\hat{\lambda}}) \;=\; S_T + \frac{1}{\gamma_2}\left[\log(\hat{y}) - \int_0^T \frac{1}{2}\hat{\lambda}_s^2 ds - \int_0^T \hat{\lambda}_s dW_s\right] \tag{77}$$

is optimal for an appropriate choice of $\hat{y}$, to be determined below. We have to check that the conditions (58) and (59) are satisfied with the stochastic processes $\hat{Z}$ and $\hat{Y}$ defined by $\hat{Z}_t = \hat{z}Z_t^{\hat{\lambda}}$ and $\hat{Y}_t = \hat{y}Z_t^{\hat{\lambda}}$, for some values of $\hat{z}, \hat{y}$. We choose $\hat{z}$ so that the IR constraint (60) is satisfied, that is

$$\hat{z} \;=\; -\gamma_1 R \quad . \tag{78}$$

Since we are taking $\hat{a} \equiv 1$, in order to satisfy (58) we need to have

$$\begin{aligned} S_T &= I_1(\hat{z}Z_T^{\hat{\lambda}}) + I_2(\hat{y}Z_T^{\hat{\lambda}}) + T/2 \\ &= -\frac{1}{\gamma_1}\log(\hat{z}) - \frac{1}{\gamma_1}\log(Z_T^{\hat{\lambda}}) - \frac{1}{\gamma_2}\log(\hat{y}) - \frac{1}{\gamma_2}\log(Z_T^{\hat{\lambda}}) + \frac{T}{2}. \end{aligned}$$

Then,

$$\begin{aligned} S_0 + T + W_T &= -\frac{1}{\gamma_1}\log(\hat{z}) - \frac{1}{\gamma_2}\log(\hat{y}) - \Gamma\log(Z_T^{\hat{\lambda}}) + \frac{T}{2} \\ &= -\frac{1}{\gamma_1}\log(\hat{z}) - \frac{1}{\gamma_2}\log(\hat{y}) + W_T + \frac{1}{2}\frac{1}{\Gamma}T + \frac{T}{2}, \end{aligned}$$

or equivalently

$$S_0 + \frac{T}{2} - \hat{\lambda}\frac{T}{2} = -\frac{1}{\gamma_1}\log(\hat{z}) - \frac{1}{\gamma_2}\log(\hat{y}) \quad . \tag{79}$$

It is now easily verified that (59) is also satisfied. The maximum utility is given by

$$E\left[U_2\left(S_T - I_2(\hat{y}Z_T^{\hat{\lambda}})\right)\right] = -\frac{\hat{y}}{\gamma_2} \quad , \tag{80}$$

where $\hat{y}$ is determined from (79), with $\hat{z}$ determined from (78).

Let us now compare this to the utility obtained from the optimal contract among those contracts which depend only on S, not on W. By Holmstrom and Milgrom [20] (see also Schättler and Sung [37]), the optimal contract of such a form is linear, and given by

$$f(S_T) = c + bS_T \quad ,$$

where

$$b = \frac{1+\gamma_2}{1+\gamma_1+\gamma_2} \quad ,$$

and c is chosen so that the IR constraint is satisfied. In this case the optimal control is $\hat{a} = b$

and the IR constraint gives

$$c = -\frac{1}{\gamma_1}\log(-\gamma_1 R) - bS_0 + \frac{b^2 T}{2}(\gamma_1 - 1) \quad . \tag{81}$$

The maximum principal's utility with this contract is

$$E\left[U_2(S_T - (c + bS_T))\right] = -\frac{1}{\gamma_2}\exp\left\{-\gamma_2\left[-c + (1-b)\left(S_0 + bT - \frac{1}{2}\gamma_2(1-b)T\right)\right]\right\} \quad .$$

Comparing this to the utility from (80), we can check that the difference of the utility from the generally optimal contract $\hat{P}_T = S_T - I_2(\hat{y}Z_T^{\hat{\lambda}})$ and the utility from the optimal S-based contract is strictly positive.

We see that in this example the first best contract $\hat{P}_T$ is not implementable with S-based contracts.

5.1.3 Example: Volatility constraints

Consider again Problem II. Using our method, it can be shown that in the model

$$dS_t = \alpha\sigma_t dt + \sigma_t W_t,$$

where W is a one-dimensional Brownian Motion, if the principal is risk-neutral, $U_2(x) = x$, and the agent has no penalty on volatility, then the principal can attain infinite utility by forcing the agent to apply infinite volatility, $\hat{\sigma} \equiv \infty$. This may not be true if there are constraints on volatility, of the type $\sigma_t \in K$ for all t, where K is a closed and convex set. We could still use the approach of Example 5.1.2, with

$$\tilde{g}(t,\lambda) = \max_{\sigma \in K}\{-g(t,\sigma) - f(t)\sigma'\lambda\} \quad . \tag{82}$$

In the one dimensional case, we consider the constraint

$$b \ \le \ \sigma_t \ \le \ c$$

for some constants b, c (the example can be extended to time-dependent bounds b_t, c_t). For simplicity, we assume that the cost function is zero.

Since our method cannot deal directly with the risk-neutral utility function $U(x) = x$, we assume that the utility functions are exponential:

$$U_i(x) \ = \ \frac{1}{\gamma_i} - \frac{1}{\gamma_i} e^{-\gamma_i x} \quad .$$

Note that if we let γ_i go to zero, then $U_i(x)$ goes to x, and we get the risk-neutral case in the limit. We can now verify, as in the previous examples, that the optimal volatility $\hat{\sigma}$ is given by

$$\hat{\sigma} \ = \ \begin{cases} b & ; \quad \text{if } \Gamma\alpha < b \\ c & ; \quad \text{if } \Gamma\alpha > c \\ \Gamma\alpha & ; \quad \text{if } b \le \Gamma\alpha \le c \end{cases}$$

where Γ is defined in (71). Finally, note that if either the principal or the agent is risk-neutral, that is, if either γ_1 or γ_2 tends to zero, then Γ tends to infinity, which implies that

$$\hat{\sigma} = c \quad ,$$

that is, the agent will optimally choose the maximum possible volatility.

6 Conclusions

In this paper we consider continuous-time, first-best risk-sharing problems, in which one of the individuals (that we call the agent, although there are no agency costs) can control both the diffusion (project selection) and the drift term (effort choice) of the underlying process, with full information for the other individual (the principal). We only consider a type of contracts (widely used in practice) in which the optimal risk-sharing rule depends on the comparison between a function of the underlying process and a benchmark. In the case of portfolio management and complete markets, the optimal contract is (ex-post) a function of the terminal value of the controlled process, and this function may be nonlinear if the principal and agent have different utility functions. Even when the contract is ex-post linear, if the agent controls the drift separately from the diffusion term, there may exist no optimal contract which is path independent ex-ante. If the agent does not control the drift independently of the diffusion term, an optimal contract can be offered as a function of the terminal value of the controlled process if a certain differential equation has a solution. Call option-type contracts are optimal for a specific choice of utility functions. The contract is linear when the agent and the principal both have exponential utility functions, or if they have the same power utility functions. When it is costly for the agent to control the diffusion term, the optimal contract is obtained by considering a fixed point of a map between the principal's optimization problem and the agent's optimization problem, or by considering a solution to a corresponding dual problem.

There are several directions in which we could extend this work. Here, the agent and the principal have the same information, and the solution is the first-best. It would be of considerable interest to study the problem with asymmetric or incomplete information, even further than what is accomplished in Williams [40] or Cvitanić, Wan and Zhang [10], by modeling adverse selection, for example. More realistic cases could also be considered, such as a possibility for the agent to cash in the contract at a random time, or the case when the time horizon is also a part of the contract. We leave these problems for future research.

7 Appendix

Proof of Theorem 3.1: The following argument is similar to standard arguments in the modern approach to the Merton's portfolio optimization problem, and due originally, in a somewhat different form, to Pliska [34], Cox and Huang [5] and Karatzas, Lehoczky and Shreve [22] (see, for example, Karatzas and Shreve [24] for a general theory, or Cvitanić and Zapatero [12] for the simpler, complete models case). For a heuristic derivation, see the proof of Proposition 3.1 below.

We only provide the proof with the agent's Problem I, as the Problem II is similar, noting that the maximum of $\beta\delta a - G(t, a)$ is attained at $\hat{a} \equiv J(t, \delta\beta)$. Moreover, in the proof of Theorem 5.1 for the general case, we deal with Problem II.

Define the dual function

$$U_2^*(y, \lambda) \; := \; \max_{s,p} \{U_2(\beta s - p) + \lambda U_1(p) - y\beta s\} \tag{83}$$

for those values of $\lambda > 0$ and y for which this is finite, constituting the effective domain $\tilde{D}_2$ of U_2^*. Similarly, define the dual function

$$V_2^*(t, \bar{y}, \lambda) \; := \; \max_{D,q} \{V_2(t, \kappa D - q) + \lambda V_1(t, q) - \bar{y}\kappa D\} \, . \tag{84}$$

We assume, for simplicity, that $V_2^*(t, \cdot, \cdot)$ and $U_2^*(\cdot, \cdot)$ are defined on the same domain $\tilde{D}_2$ which does not depend on t.

We show in the lemma below that the maximums are attained at

$$\beta\hat{s} - \hat{p} = I_2^U(y) \; ; \quad \hat{p} = I_1^U(y/\lambda) \quad , \tag{85}$$

$$\kappa\hat{D} - \hat{q} = I_2^V(t, \bar{y}) \; ; \quad \hat{q} = I_1^V(t, \bar{y}/\lambda) \quad . \tag{86}$$

Set now $s = S_T$ and $p = P_T$ in the definition of U_2^*. This, together with the IR constraint, gives us

$$E\left[U_2(\beta S_T - P_T) + \int_0^T V_2(s, \kappa D_s - q_s)ds\right] \tag{87}$$
$$\leq \; E\left[U_2^*(yZ_T, \lambda) + \int_0^T V_2^*\left(s, \frac{\beta}{\kappa}yZ_s, \lambda\right)ds\right] + yE\left[\beta Z_T S_T + \beta\int_0^T Z_s D_s ds\right]$$
$$-\lambda R - \lambda E\left[\int_0^T G(s, a_s)ds\right] \quad ,$$

where R is the reservation utility for the agent and y is a constant. Define now the dual function

$$\tilde{G}(t, z) := \max_a\{\beta\delta a z - \lambda G(t, a)\} \tag{88}$$

for which, if $z = yZ_t$, the maximum is attained at

$$\hat{a}_t = J\left(t, \beta\delta\frac{y}{\lambda}Z_t\right). \tag{89}$$

Using $\tilde{G}(t, yZ_t) \geq -\lambda G(t, a_t) + \beta\delta a_t y Z_t$ and (10) we get the upper bound for the principal's problem:

$$E\left[U_2(\beta S_T - P_T) + \int_0^T V_2(s, \kappa D_s - q_s)ds\right] \tag{90}$$
$$\leq \; E\left[U_2^*(yZ_T, \lambda) + \int_0^T V_2^*\left(s, \frac{\beta}{\kappa}yZ_s, \lambda\right)ds\right] + y\beta S_0 + \int_0^T \tilde{G}(s, yZ_s)ds - \lambda R \quad .$$

The upper bound is attained if (89) is satisfied, and, by (85), (86), if

$$\beta S_T - P_T \;=\; I_2^U(yZ_T) \quad , \tag{91}$$

$$P_T \;=\; I_1^U(yZ_T/\lambda) \quad , \tag{92}$$

$$\kappa D_t - q_t \;=\; I_2^V\left(t, \frac{\beta}{\kappa}yZ_t\right) \quad , \tag{93}$$

$$q_t \;=\; I_1^V\left(t, \frac{\beta}{\kappa\lambda}yZ_t\right) \quad , \tag{94}$$

if the IR constraint is satisfied as equality:

$$R = E\left[U_1\left(I_1^U(yZ_T/\lambda)\right) + \int_0^T \left\{V_1\left(t, I_1^V\left(t, \frac{\beta}{\kappa\lambda}yZ_t\right)\right) - G\left(t, J\left(t, \beta\delta\frac{y}{\lambda}Z_t\right)\right)\right\} dt\right] , \tag{95}$$

and if the martingale property (10), $E[M_T^S] = S_0$ holds, or equivalently,

$$S_0 = E\left[Z_T S_T + \int_0^T Z_s \left\{D_s - \delta J\left(s, \beta\delta\frac{y}{\lambda}Z_s\right)\right\} ds\right] . \tag{96}$$

Equations (91)-(94) and (96) imply that the number $y = \hat{y}$ has to be chosen so that the *principal's feasibility constraint*

$$\beta S_0 + E\left[\int_0^T \beta\delta Z_t J\left(t, \beta\delta\frac{\hat{y}}{\lambda}Z_t\right) dt\right] \tag{97}$$

$$= E\left[Z_T\left\{I_1^U(\hat{y}Z_T/\lambda) + I_2^U(\hat{y}Z_T)\right\} + \int_0^T Z_s\frac{\beta}{\kappa}\left[I_1^V\left(s, \frac{\beta}{\kappa\lambda}\hat{y}Z_s\right) + I_2^V\left(s, \frac{\beta}{\kappa}\hat{y}Z_s\right)\right] ds\right]$$

is satisfied. If we set $\lambda = \hat{y}/\hat{z}$, such $\hat{y}$ exists by Assumption 3.3, and the only remaining thing we have to check is whether there exists a process σ so that at the final time we have

$$\beta S_T = \beta\hat{S}_T = I_1^U(\hat{z}Z_T) + I_2^U(\hat{y}Z_T) . \tag{98}$$

This follows from the Martingale Representation Property, that is, from Assumption 3.1.

◇

Lemma 7.1 *The values* $\hat{s}, \hat{p}, \hat{D}, \hat{q}$ *from* (85), (86) *are optimal for the maximization in the definition of* U_2^*, V_2^*.

Proof: We only show the case for U_2^*. Values $\hat{s}, \hat{p}$ are determined so that they make the partial derivatives equal to zero. Thus, it only remains to show that the Hessian of the problem is a negative definite matrix. Let us introduce the change of variables

$$\tilde{s} = \beta s \quad .$$

Denote

$$f(\tilde{s}, p) \;=\; U_2(\tilde{s} - p) - y\tilde{s} + \lambda U_1(p).$$

We compute the second derivatives as follows:

$$\begin{aligned}
\frac{\partial^2 f(\tilde{s}, p)}{\partial \tilde{s}^2} &= U_2''(\tilde{s} - p), \\
\frac{\partial^2 f(\tilde{s}, p)}{\partial p^2} &= U_2''(\tilde{s} - p) + \lambda U_1''(p) \quad , \\
\frac{\partial^2 f(\tilde{s}, p)}{\partial p \partial \tilde{s}} &= -U_2''(\tilde{s} - p).
\end{aligned}$$

Then we find that, for a given vector $(a, b)' \neq (0, 0)'$ and the Hessian matrix $H(\tilde{s}, p)$, we have

$$(a, b)H(\tilde{s}, p)(a, b)' \;=\; U_2''(\tilde{s} - p)(a - b)^2 + b^2 \lambda U_1''(p).$$

Since $\lambda > 0$, and since U_2, U_1 are strictly concave, the right-hand side is negative.

$\diamond$

Proof of Proposition 3.1: Again, we only consider Problem I. We first develop some heuristics for finding the optimal strategy of the agent, while a rigorous proof is given after that. We can consider the fact that the process M^S is a martingale, and in particular that $E[M_T^S] = S_0$, to be a constraint on the agent's problem; see (10). Thus, the agent has to find controls a, σ, D that maximize

$$E\left[U_1\left(\beta S_T - I_2^U(\hat{y}Z_T)\right) + \int_0^T \left\{V_1\left(s, \kappa D_s - I_2^V\left(s, \frac{\beta}{\kappa}\hat{y}Z_s\right)\right) - G(s, a_s)\right\} ds\right]$$

$$-\beta z_A \left(E \left[Z_T S_T - \delta \int_0^T Z_s \hat{a}_s ds + \int_0^T Z_s D_s ds \right] - S_0 \right)$$

where z_A is a Lagrange multiplier. Taking derivatives with respect to a_s, S_T, D_s and setting them equal to zero (and neglecting the expectation) in the previous maximization problem, we conjecture that it is optimal to choose the controls $\hat{a}, \hat{\sigma}, \hat{D}$ so that

$$\hat{a}_t \;=\; J(t, \beta\delta z_A Z_t) \quad , \tag{99}$$

$$\beta S_T - I_2^U(\hat{y} Z_T) \;=\; I_1^U(z_A Z_T) \tag{100}$$

and

$$\kappa \hat{D}_t - I_2^V\left(t, \frac{\beta}{\kappa}\hat{y} Z_t\right) \;=\; I_1^V\left(t, \frac{\beta}{\kappa} z_A Z_t\right). \tag{101}$$

If we substitute this into the martingale property (10), $E[M_T^S] = S_0$, we get that the number z_A has to satisfy the *agent's feasibility constraint*

$$\beta S_0 = E\left[Z_T \left\{ I_1^U(z_A Z_T) + I_2^U(\hat{y} Z_T) \right\} - \delta\beta \int_0^T Z_s \hat{a}_s ds + \beta \int_0^T Z_s \hat{D}_s ds \right] \quad . \tag{102}$$

In order to provide a rigorous proof of (100) and (101), we need to introduce *dual functions* $\tilde{U}_1$, $\tilde{V}_1$ of U_1, V_1 defined by

$$\tilde{U}_1(z) \;:=\; \max_x \{U_1(x) - xz\} \quad , \qquad \tilde{V}_1(t,z) \;:=\; \max_y \{V_1(t,y) - yz\} \quad ,$$

and

$$\tilde{G}(t,z) \;:=\; \max_a \{\beta\delta a z - G(t,a)\}.$$

The domain $\tilde{D}_1$ of $\tilde{U}_1$ consists of the values of z for which $\tilde{U}_1(z) < \infty$. We assume for simplicity that $\tilde{V}_1(t,\cdot)$ has the same domain, for all t. Note that performing the maximization in the definitions of $\tilde{U}_1$, $\tilde{V}_1$, $\tilde{G}$, we get that the optimal x, y, a are given by

$$\hat{x} = I_1^U(z) \quad , \hat{y} = I_1^V(t,y) \quad , \hat{a} = J(t, \beta\delta z) \quad , \tag{103}$$

where I_1^U, $I_1^V(t,\cdot)$, $J(t,\cdot)$ are the inverse functions of U_1', $\frac{\partial}{\partial y}V_1(t,\cdot)$, $\frac{\partial}{\partial a}G(t,\cdot)$.

By the definition of the dual functions we have

$$
\begin{aligned}
& E\left[U_1\left(\beta S_T - I_2^U(\hat{y}Z_T)\right) + \int_0^T \left\{V_1\left(s, \kappa D_s - I_2^V\left(\frac{\beta}{\kappa}\hat{y}Z_s\right)\right) - G(s,a_s)\right\} ds\right] \quad (104)\\
\leq \; & E\left[\tilde{U}_1(z_A Z_T) + \int_0^T \left\{\tilde{V}_1\left(s, \frac{\beta}{\kappa} z_A Z_s\right) - G(s,a_s)\right\} ds\right] \\
& + z_A E\left[Z_T\left(\beta S_T - I_2^U(\hat{y}Z_T)\right) + \int_0^T Z_s \left[\beta D_s - \frac{\beta}{\kappa} I_2^V\left(s, \frac{\beta}{\kappa}\hat{y}Z_s\right)\right] ds\right].
\end{aligned}
$$

From $E[M_T^S] = S_0$ we get

$$
E\left[S_T Z_T + \int_0^T Z_s D_s ds\right] \; = \; S_0 + \delta E\left[\int_0^T Z_s \hat{a}_s ds\right],
$$

and using the definition of $\tilde{G}$, from (104) we get the following upper bound on the value of the agent's optimization problem:

$$
\begin{aligned}
& E\left[\tilde{U}_1(z_A Z_T) + \int_0^T \left\{\tilde{V}_1\left(s, \frac{\beta}{\kappa} z_A Z_s\right) + \tilde{G}(s, z_A Z_s)\right\} ds\right] \\
& \quad + z_A \beta S_0 - z_A E\left[Z_T I_2^U(\hat{y}Z_T) + \int_0^T Z_s \frac{\beta}{\kappa} I_2^V\left(s, \frac{\beta}{\kappa}\hat{y}Z_s\right) ds\right].
\end{aligned}
$$

By (103), this upper bound is attained if (99), (100) and (101) are satisfied, where z_A is chosen so that (102) is satisfied. With this, and comparing the feasibility constraints (97) and (102), we see that we need to take $z_A = \hat{y}/\lambda$, and thus, by choosing exactly the controls $\hat{a}, \hat{\sigma}, \hat{D}$ which are the first-best, the agent will attain the upper bound for her utility.

◇

Proof of Proposition 3.2: Given such a contract it can be seen similarly as in the proof of Proposition 3.1 that the agent will optimally act so that the process S satisfies

$$
Z_t S_t \; := \; E_t[Z_T s(z_d Z_T)] \quad . \qquad (105)
$$

and in particular

$$S_T = s(z_d Z_T)$$

where the function $s(z)$ is determined from (33). From (33) and (31), we get

$$U_2'(\beta S_T - d(S_T)) \;=\; \hat{y} Z_T$$

or equivalently

$$\beta S_T - d(S_T) \;=\; I_2(\hat{y} Z_T).$$

But then the principal's utility is equal to

$$E[U_2(\beta S_T - P_T)] \;=\; E[U_2(I_2(\hat{y} Z_T))],$$

which is optimal.

◇

Proof of Corollary 3.1: Inspecting the proof of Theorem 3.1, but with $\delta = D = q = 0$, we see that any contract satisfying (35) attains the principal's upper bound. Moreover, that proof also shows that this equality has to be satisfied for the principal to attain his maximum utility.

◇

Proof of Theorem 3.2: Suppose that the contract is offered in the form $P_T = f(S_T) = I_1(\hat{z} h(S_T))$, assumed to be linear. Given this contract, similarly as in the proof of Proposition 3.1, we can see that the agent would choose $\sigma = \tilde{\sigma}$ so that

$$P_T = I_1(z^f Z_T)$$

for some z^f. Denote by $\tilde{S}$ the corresponding process S. We also have $P_T = I_1(\hat{z}h(\tilde{S}_T))$, thus

$$z^f Z_T = \hat{z}h(\tilde{S}_T). \tag{106}$$

Note that $\tilde{S}_T = f^{-1}(I_1(z^f Z_T))$, and by the martingale property,

$$S_0 \ = \ E[Z_T\tilde{S}_T] \ = \ E\left[Z_T f^{-1}(I_1(z^f Z_T))\right].$$

This means that $z^f = z^*$, the unique solution to the above equation. Denote now by $\hat{S}$ the process S corresponding to some optimal contract. From Corollary 3.1, any such contract satisfies $\beta\hat{S}_T = I_1(\hat{z}Z_T) + I_2(\hat{y}Z_T)$, which means that $\hat{S}_T = f^{-1}(I_1(\hat{z}Z_T))$. By the martingale property again, we have

$$S_0 \ = \ E[Z_T\hat{S}_T] \ = \ E\left[Z_T f^{-1}(I_1(\hat{z}Z_T))\right].$$

This means that also $\hat{z} = z^*$, thus $\hat{z} = z^f$, and from (106), $h(\tilde{S}_T) = Z_T$. Hence, we get

$$\beta\tilde{S}_T - P_T \ = \ \beta\tilde{S}_T - I_1(\hat{z}Z_T) = I_2(\hat{y}h(\tilde{S}_T)).$$

Therefore, the principal's utility when offering the contract $f(S_T)$ is

$$E\left[U_2(I_2(\hat{y}h(\tilde{S}_T)))\right] \ = \ E\left[U_2(I_2(\hat{y}Z_T))\right],$$

which is optimal.

$\diamond$

Computations for Example 4.2: Similarly to Proposition 3.1, the agent will choose the control $a = \tilde{a}$ given by

$$\tilde{a}_t = J(\delta b).$$

Using the method of that proof, we can also see that he will choose $\sigma = \tilde{\sigma}$ so that

$$c + bS_T - TG(J(\delta b)) \;=\; -\frac{1}{\gamma_1}\log(\tilde{z}Z_T) \tag{107}$$

where $\tilde{z}$ is determined from the agent's feasibility (martingale) condition

$$c + bS_0 - TG(J(\delta b)) + T\delta bJ(\delta b) \;=\; -\frac{1}{\gamma_1}(\log(\tilde{z}) + \alpha^2 T/2). \tag{108}$$

Here, we use the fact that

$$E\left[Z_T\log(Z_T)\right] \;=\; E\left[Z_T(-\alpha^2T/2 - \alpha W_T)\right] \;=\; -\alpha^2T/2 - \alpha E[Z_TW_T] \;=\; \alpha^2T/2. \tag{109}$$

Now, the utility of the principal is, substituting S_T from (107),

$$\begin{aligned} E\left[U_2(\beta S_T - P_T)\right] &= E\left[U_2((\beta - b)S_T - c)\right] \\ &= E\left[U_2\left(\left(\frac{\beta}{b} - 1\right)\left[TG(J(\delta b)) - \frac{1}{\gamma_1}\log(\tilde{z}Z_T)\right] - c\frac{\beta}{b}\right)\right] \end{aligned} \tag{110}$$

Substituting for the utility function and the values a, b from (37), we get

$$\begin{aligned} E[U_2(\beta S_T - P_T)] &= E\Big[U_2\Big(\left(\frac{\gamma_1}{\gamma_2}\right)\left[TG(J(\delta b)) - \frac{1}{\gamma_1}\log(\tilde{z}Z_T)\right] \\ &\qquad - \left(\frac{\gamma_1}{\gamma_2}\right)\left[TG(J(\delta\beta)) + \frac{1}{\gamma_1}\log(\hat{y}/\hat{z})\right]\Big)\Big] \\ &= E\left[U_2\left(\left(\frac{\gamma_1}{\gamma_2}\right)[TG(J(\delta b)) - TG(J(\delta\beta))] - \frac{1}{\gamma_2}\log\left(\frac{\tilde{z}\hat{y}}{\hat{z}}Z_T\right)\right)\right] \\ &= -\frac{\hat{y}}{\gamma_2}\exp\left\{-\gamma_1\left[TG(J(\delta b)) - TG(J(\delta\beta)) + \frac{1}{\gamma_1}\log(\hat{z}/\tilde{z})\right]\right\}. \end{aligned} \tag{111}$$

On the other hand, we know that the principal's utility given the optimal contract $\hat{P} = \beta S_T - I_2(\hat{y}Z_T)$ is

$$E[U_2(\beta S_T - \hat{P}_T)] \;=\; E[U_2(I_2(\hat{y}Z_T))] \;=\; -\frac{\hat{y}}{\gamma_2}.$$

Thus, in order for the contract $f(S_T)$ to be optimal, the exponent in (111) should be equal to zero. In order to compute this exponent, we substitute $\tilde{z}$ from (108), we substitute b, c from (37), and we also use the principal's feasibility (martingale) condition (97) (with $\lambda = \hat{y}/\hat{z}$) for the optimal contract:

$$\begin{aligned} \beta S_0 &= -\frac{1}{\gamma_1} E\left[Z_T \log(\hat{z} Z_T)\right] - \frac{1}{\gamma_2} E\left[Z_T \log(\hat{y} Z_T)\right] + TG(J(\delta\beta)) - T\delta\beta J(\delta\beta) \\ &= -\frac{1}{\gamma_1}\log(\hat{z}) - \frac{1}{\gamma_2}\log(\hat{y}) + TG(J(\delta\beta)) - T\delta\beta J(\delta\beta) - \frac{\alpha^2 T}{2}(\gamma_1^{-1} + \gamma_2^{-1}). \quad (112) \end{aligned}$$

Doing that we obtain from (112)

$$\begin{aligned} & TG(J(\delta b)) - TG(J(\delta\beta)) + \frac{1}{\gamma_1}\log(\hat{z}/\tilde{z}) \\ = \;& TG(J(\delta b)) - TG(J(\delta\beta)) + \frac{1}{\gamma_1}\log(\hat{z}) + c + bS_0 - TG(J(\delta b)) + T\delta b J(\delta b) + \frac{1}{\gamma_1}\alpha^2\frac{T}{2} \\ = \;& TG(J(\delta b)) - TG(J(\delta\beta)) + \frac{1}{\gamma_1}\log(\hat{z}) + \frac{\gamma_2^{-1}}{\gamma_2^{-1}+\gamma_1^{-1}}\left[TG(J(\delta\beta)) + \gamma_1^{-1}\log(\hat{y}/\hat{z})\right] \\ & +\beta\frac{\gamma_1^{-1}}{\gamma_2^{-1}+\gamma_1^{-1}}S_0 - TG(J(\delta b)) + T\delta b J(\delta b) + \frac{1}{\gamma_1}\alpha^2\frac{T}{2} \\ = \;& \frac{\gamma_1^{-1}}{\gamma_2^{-1}+\gamma_1^{-1}}\left(T\delta b J(\delta b) - T\delta\beta J(\delta\beta)\right). \end{aligned}$$

Since b is strictly smaller than β, the exponent in (111) is not zero, and the principal's utility when giving the contract $f(S_T)$ is smaller than giving the optimal contract $\hat{P}_T = \beta S_T - I_2(\hat{y} Z_T)$.

◇

Computations for Example 4.3: Consider maximizing the function

$$F(s) := U_1(d(s)) - zs = \log(d(s)) - zs \quad ,$$

for $z > 0$. Note that by L'Hospital's rule

$$\lim_{s\to\pm\infty} \frac{\log(d(s))}{s} = \lim_{s\to\pm\infty} \frac{d'(s))}{d(s)} = \lim_{s\to\pm\infty} \frac{z_d}{\hat{y}} e^{-\beta s + d(s)} \quad .$$

Thus, the limit at $s = -\infty$ is ∞ and the limit at $s = \infty$ is zero, if $C < 0$. This implies that

$$F(-\infty) = -\infty \qquad \text{and} \qquad F(\infty) = -\infty \quad .$$

Therefore, if there is a unique value $s = s(z)$ for which the first derivative of the function F is zero, then the maximum is attained at that value. This is equivalent to finding a unique value $s(z)$ such that

$$\frac{z_d}{\hat{y}} G(s(z)) - z = 0$$

where

$$G(s) = e^{-\beta s + d(s)} \quad . \tag{113}$$

Note that, if $C < 0$,

$$G(-\infty) = \infty \qquad \text{and} \qquad G(\infty) = 0 \quad .$$

Therefore, it is sufficient to show that $G'(s)$ is always negative, which is equivalent to showing that

$$d'(s) < \beta \quad . \tag{114}$$

In order to show this, note that

$$d'(s) = (Ei^{-1})'(x(s) + C)\beta x(s), \quad \text{where } x(s) = -\frac{z_d}{\beta \hat{y}} e^{-\beta s} < 0 \quad .$$

It is easily verified that the maximum of $d'(s)$ over $C \leq 0$ is attained at $C = 0$. Thus, from the last equation, in order to prove (114), we need to show that

$$(Ei^{-1})'(x)x < 1$$

for $x < 0$. This is equivalent to

$$Ei'(Ei^{-1}(x)) < x \quad .$$

By transforming the variables as

$$x = Ei(y)$$

the last inequality is equivalent to

$$Ei'(y) < Ei(y)$$

or

$$Ei(y) > e^y/y, \quad y < 0 \quad .$$

Finally, by integration by parts, we have

$$Ei(y) \; = \; \int_{-\infty}^{y} \frac{e^t}{t} dt \; = \; \frac{e^y}{y} + \int_{-\infty}^{y} \frac{e^t}{t^2} dt \; > \; \frac{e^y}{y} \quad .$$

Thus, we have shown that (114) holds and so there is a unique value $s(z)$ that maximizes $U_1(d(s)) - sz$.

It remains to show that $z_d > 0$ and $C < 0$ can be chosen so that

$$E[Z_T s(z_d Z_T)] = S_0 \qquad \text{and} \qquad E[U_1(d(S_T))] = R \quad , \tag{115}$$

where the process S is defined by (105). Recall that

$$s(z) = G^{-1}_{z_d,C}(z\hat{y}/z_d)$$

where the function $G_{z_d,C}$ is defined in (113). Hence, we need to have

$$E[Z_T G^{-1}_{z_d,C}(\hat{y} Z_T)] = S_0 \quad . \tag{116}$$

From the definition of the function $d(\cdot)$, it can be seen that the left-hand side above covers all values $S_0 \in (-\infty, \infty)$ as z_d ranges through $(0, \infty)$ for fixed $C < 0$, and similarly, as C ranges through $(-\infty, 0)$ for fixed $z_d > 0$. The second condition in (115) becomes

$$E\left[\log\left(d_{z_d,C}\left(G^{-1}_{z_d,C}(\hat{y}Z_T)\right)\right)\right] = R \quad .$$

Again, it can be seen that the left-hand side above covers all values $R \in (-\infty, \infty)$ as z_d ranges through $(0, \infty)$ for fixed $C < 0$, and as C ranges through $(-\infty, 0)$ for fixed $z_d > 0$. We conclude that there exist values $z_d > 0, C < 0$ so that (115) holds.

$\diamond$

Proof of Theorem 5.1: We present an extension of the argument for the complete model case, but this time we do it for Problem II only. Consider first the first-best solution, that is, the problem of the principal maximizing her utility, under the IR constraint for the principal. Recall the dual function

$$U_2^*(y, \lambda) \;:=\; \max_{s,p}\{U_2(\beta s - p) + \lambda U_1(p) - y\beta s\} \tag{117}$$

for $\lambda > 0$, and y in the effective domain of U_2^*. Also recall that the maximum is attained at

$$\beta\hat{s} - \hat{p} = I_2^U(y)\;; \quad \hat{p} = I_1^U(y/\lambda) \quad . \tag{118}$$

Set now $\beta s = \beta S_T - \int_0^T \bar{G}(t, a_t, \sigma_t, S_t)dt$ and $p = P_T - \int_0^T \bar{G}(t, a_t, \sigma_t, S_t)dt$ in (64). Then, we have

$$\begin{aligned} U_2^*(\hat{Z}_T, \lambda) \;\geq\; & U_2(\beta S_T - P_T) + \lambda U_1\left(P_T - \int_0^T \bar{G}(t, a_t, \sigma_t, S_t)dt\right) \\ & -\hat{Z}_T\left(\beta S_T - \int_0^T \bar{G}(t, a_t, \sigma_t, S_t)dt\right) . \end{aligned}$$

Taking expectations, we then obtain

$$E\left[U_2(\beta S_T - P_T)\right] \;\le\; E\left[U_2^*(\hat{Z}_T, \lambda)\right] + L(\hat{Z}) - \lambda R \;\;, \tag{119}$$

where R is the reservation utility for the agent. Note that for a given λ, this is an upper bound on the principal's utility. By (118), the upper bound is attained if (58) – (61) are satisfied, if we set $\lambda = 1/\bar{c}$. Thus, $\hat{a}, \hat{\sigma}$ are optimal for the principal's first-best problem.

Let us now take into consideration the agent's problem. Suppose that the principal offers the agent contract (61). Then, by the definition of $\tilde{U}_1$ (see (54)), we have the following upper bound on the agent's utility, for any stochastic process $Y \in \mathcal{D}_L$:

$$\begin{aligned} E\left[U_1\left(\hat{P}_T - \textstyle\int_0^T \bar{G}(t, a_t, \sigma_t, S_t)dt\right)\right] &= E\left[U_1\left(\beta S_T - I_2(\hat{Z}_T) - \textstyle\int_0^T \bar{G}(t, a_t, \sigma_t, S_t)dt\right)\right] \\ &\le E\left[\tilde{U}_1(Y_T) + L(Y) - Y_T I_2(\hat{Z}_T)\right] \;\;. \end{aligned} \tag{120}$$

Moreover, the smallest such bound is obtained by taking the infimum of the right-hand side, and it is attained at $Y = \bar{c}\hat{Z}$, by definition of $\hat{Z}$.

This smallest upper bound will be attained if (58) is satisfied and if (59) is satisfied with $\hat{Z}$ replaced by $\bar{c}\hat{Z}$. By (60), this will indeed be satisfied if the agent uses the first-best controls $\hat{a}, \hat{\sigma}$, because $L(\bar{c}\hat{Z}) = \bar{c}L(\hat{Z})$.

The last statement of the theorem is true because, with $P_T = \hat{P}_T$ and with S_T corresponding to $(\hat{a}, \hat{\sigma})$, inequality (119) is satisfied by all $\lambda > 0$, $Z \in \mathcal{D}_L$, and it is satisfied as equality by $\lambda = \frac{1}{\bar{c}}$, $Z = \hat{Z}$.

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References

[1] A. Amershi, J. Stoeckenius, The theory of syndicates and linear sharing rules, Econometrica 51 (1983), 1407–1416.

[2] K. Arrow, The role of securities in the optimal allocation of risk bearing, Colloques Internationaux du Centre National de la Recherche Scientifique 50 (1953), 41–49.

[3] P. Bolton, C. Harris, The continuous-time principal-agent problem: Frequent-monitoring contracts, unpublished manuscript, Princeton University, 2001.

[4] K. Borch, Equilibrium in a reinsurance market, Econometrica 30 (1962), 424–444.

[5] J. Cox, C.-f. Huang, Optimal consumption and portfolio policies when asset prices follow a diffusion process, J. Econ. Theory 49 (1989), 33–83.

[6] D. Cuoco, J. Cvitanić, Optimal consumption choices for a large investor, J. Econ. Dynam. Control 22 (1998), 401–436.

[7] J. Cvitanić, Nonlinear financial markets: Hedging and portfolio optimization, in: M. Dempster, S. Pliska, (Eds.) Mathematics of Derivative Securities: Proceedings of the Isaac Newton Institute, Cambridge University Press, Cambridge, 1997.

[8] J. Cvitanić, I. Karatzas, Convex duality in constrained portfolio optimization, Annals of Applied Probability 2 (1992), 767–818.

[9] J. Cvitanić, W. Schachermayer, H. Wang, Utility maximization in incomplete markets with random endowment, Finance & Stochastics 5 (2002), 259–272.

[10] J. Cvitanić, X. Wan, J. Zhang, Optimal contracts in continuous-time models, unpublished manuscript, Caltech 2005a.

[11] J. Cvitanić, X. Wan, J. Zhang, Continuous-time principal-agent problems with hidden action: The weak formulation, unpublished manuscript, Caltech 2005b.

[12] J. Cvitanić, F.Zapatero, Introduction to the Economics and Mathematics of Financial Markets, MIT Press, Cambridge, MA, 2004.

[13] P. DeMarzo, M. Fishman, Agency and optimal investment dynamics, unpublished manuscript, Stanford University 2003.

[14] J. Detemple, S. Govindaraj, M. Loewenstein, Optimal contracts and intertemporal incentives with hidden actions, unpublished manuscript, Boston University 2001.

[15] D. Duffie, P.Y. Geoffard, C. Skiadas, Efficient and equilibrium allocations with stochastic differential utility, J. Math. Econ. 23 (1994), 133–146.

[16] B. Dumas, R. Uppal, T. Wang, Efficient intertemporal allocations with recursive utility, J. Econ. Theory 93 (2000), 240–259.

[17] L. Epstein, The global stability of efficient intertemporal allocations, Econometrica 55 (1987), 329–355.

[18] H. He, N. Pearson, Consumption and portfolio policies with incomplete markets and short-sale constraints, J. Econ. Theory 54 (1991), 259–304.

[19] M. Hellwig, K. Schmidt, Discrete-time approximations of the Holmström-Milgrom Brownian motion model of intertemporal incentive provision, forthcoming in Econometrica (2003).

[20] B. Holmström, P. Milgrom, Aggregation and linearity in the provision of intertemporal incentives, Econometrica 55 (1987), 303–328.

[21] J. Hugonnier, D. Kramkov, Optimal investment with random endowments in incomplete markets, unpublished manuscript, Carnegie Mellon University 2002.

[22] I. Karatzas, J. Lehoczky and S. Shreve, Optimal portfolio and consumption decisions for a "small investor" on a finite horizon, SIAM Journal of Control and Optimization 25 (1987), 1557–1586.

[23] I. Karatzas, J.Lehoczky, S. Shreve, G. Xu, Martingale and duality methods for utility maximization in an incomplete market, SIAM Journal of Control and Optimization 29 (1991), 702–730.

[24] I. Karatzas, S. Shreve, Methods of Mathematical Finance, Springer-Verlag, New York, 1998.

[25] I. Karatzas, G. Žitković, Optimal consumption from investment and random endowment in incomplete semimartingale markets, unpublished manuscript, Columbia University 2002.

[26] D. Kramkov, W. Schachermayer, A condition on the asymptotic elasticity of utility functions and optimal investment in incomplete markets, Annals of Applied Probability 9 (1999), 904–950.

[27] K. Larsen, Optimal portfolio delegation when parties have different coefficients of risk aversion, forthcoming in Quantitative Finance (2005).

[28] R. Lucas, N. Stokey, Optimal growth with many consumers, J. Econ. Theory 32 (1984), 139–171.

[29] R.C. Merton, Optimum consumption and portfolio rules in a continuous-time model, J. Econ. Theory 3 (1971), 373–413.

[30] H. Müller, The first-best sharing rule in the continuous-time principal agent problem with exponential utility, J. Econ. Theory 79 (1998), 276–280.

[31] H. Müller, Asymptotic efficiency in dynamic principal-agent problems, J. Econ. Theory 91 (2000), 292–301.

[32] H. Ou-Yang, Optimal contracts in a continuous-time delegated portfolio management problem, Rev. Finan. Stud. 16 (2003), 173–208.

[33] C. Phelan, R. Towsend, Computing multi-period, information-constrained optima, Rev. Econ. Stud. 58 (1991), 853–881.

[34] S. Pliska, A stochastic model of continuous trading: optimal portfolios, Mathematics of Operations Research 11 (1986), 371–384.

[35] S. Ross, The economic theory of agency: The principal's problem, Amer. Econ. Rev. 63 (1973), 134–139.

[36] Y. Sannikov, A continuous-time version of the principal-agent problem, unpublished manuscript, UC Berkeley 2004.

[37] H. Schättler, J. Sung, The first-order approach to the continuous-time principal-agent problem with exponential utility, J. Econ. Theory 61 (1993), 331–371.

[38] S. Spear, S. Srivastava, On repeated moral hazard with discounting, Rev. Econ. Stud. 54 (1987), 599–617.

[39] J. Sung, Linearity with project selection and controllable diffusion rate in continuous-time principal-agent problems, RAND J. Econ. 26 (1995), 720–743.

[40] N. Williams, On dynamic principal-agent problems in continuous time, unpublished manuscript, Princeton University, 2005.

[41] R. Wilson, The theory of syndicates, Econometrica 36 (1968), 119–132.