

# Employee Stock Options: Accounting for Optimal Hedging, Suboptimal Exercises, and Contractual Restrictions \*

Tim Leung<sup>†</sup>

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## Abstract

Employee stock options (ESOs) have become an integral component of compensation in the U.S. In view of their significant cost to firms, the Financial Accounting Standards Board (FASB) has mandated expensing ESOs since 2004. The main difficulty of ESO valuation lies in the uncertain timing of exercises, and a number of contractual restrictions of ESOs further complicate the problem.

We present a valuation framework that captures the main characteristics of ESOs. Specifically, we incorporate the holder's risk aversion, and hedging strategies that include both *dynamic* trading of a correlated asset and *static* positions in market-traded options. Their combined effect on ESO exercises and costs are evaluated along with common features like vesting periods, job termination risk and multiple exercises. This leads to the study of a joint *stochastic control* and *optimal stopping* problem. We find that ESO values are much less than the corresponding Black-Scholes prices due to early exercises, which arise from risk aversion and job termination risk; whereas static hedges induce holders to delay exercises and increase ESO costs.

## 1 Introduction

Employee stock options are American call options granted by a firm to its employees as a form of benefit in addition to salary. An ESO entitles the employee to buy a share of the firm's stock at a specified price on or before the expiration date. Firms use these options to provide compensation

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<sup>†</sup>ORFE Department, Princeton University, E-Quad, Princeton NJ 08544; [siutang@princeton.edu](mailto:siutang@princeton.edu).

to the employees and align their incentives with the shareholders by creating a shared sense of ownership. Since the mid 1980s, stock options have become an integral component of compensation in the US. According to Hall and Murphy (2002), in fiscal year of 1999, 94% of S&P 500 companies granted options to their top executives, and the total value accounted for 47% of total pay for the CEOs. Due to the extensive use of ESOs, the Financial Accounting Standards Board (FASB) has become concerned about the cost of these options to shareholders. Since 2004, the FASB has required firms to estimate and report cost of ESOs. This gives rise to the need to create a reasonable valuation method for these options.

In order to value ESOs, it is important to understand the characteristics of ESOs, and distinguish them from market-traded options. In most cases, ESOs are not immediately exercisable - firms usually require employees to hold the options for a certain period from the grant date. This period is called the *vesting* period (typically, 2 to 4 years). During the vesting period, the employee's employment termination with the firm, voluntarily or forced, will lead to forfeiture of his option (see Figure 2.1). Moreover, the employee cannot sell the ESO or hedge it by short selling the firm's stock<sup>1</sup>, but partial hedge is possible by trading correlated assets, for example, the S&P 500 index. The inability to sell or perfectly hedge the option may induce the employee to exercise the ESO early and invest the option proceeds elsewhere. In fact, empirical studies on ESOs (see Huddart and Lang (1996) and Marquardt (2002)) have shown that employees tend to exercise their ESOs very early, often soon after the vesting period, which is deemed suboptimal by standard no-arbitrage pricing theory.

In existing literature, ESO valuation models can be categorized into risk-neutral models and utility-based models. Among risk-neutral models, one approach is to prescribe an *ad hoc* ESO exercise boundary (Hull and White (2004) and Cvitanic et al. (2004)), while another approach models option exercise as the first arrival of an exogenous counting process (Jennergren and Naslund (1993) and Carr and Linetsky (2000)). In these models, there is no optimality justification for the employee's exercise policy, and the exercise time does not interact with the employee's risk attitude, hedging strategies, or job termination risk. In Section 3, we will see that both risk aversion and job

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<sup>1</sup>According to Section 16(c) of the U.S. Securities Exchange Act, executives are precluded from short-selling the shares of their employer. The FASB statement 123R (see paragraph B80) indicates that "many public entities have established share trading policies that effectively extend that prohibition to other employees." This short sales restriction has been adopted in the literature on ESOs; e.g. Carpenter (1998).

termination risk induce the employee to adopt a more conservative exercising strategy (Figure 3.1).

Another approach, the utility-based approach, incorporates the effects of non-tradability and hedging restrictions. Detemple and Sundaresan (1999) show in a binomial model that, in the presence of hedging restrictions, a risk-averse employee may find it optimal to exercise his American-style ESO early. Part of the recent advancement in financial engineering involves the continued development of a methodology called *utility indifference pricing*. This mechanism is a dynamic version of the *certainty equivalent* concept; it accounts for the holder's investment opportunities or partial hedging instruments in addition to the option. The option holder's *utility indifference price* is defined as the amount of money that he is willing to pay so that his maximal expected utility is the same as that from an investment without the claim. Henderson (2005) studies the indifference price of a European-style ESO for an employee who dynamically trades an asset partially correlated to the firm's stock. Oberman and Zariphopoulou (2003) consider the indifference pricing of an American option with finite maturity in an incomplete market model driven by diffusion price processes.

In this paper, we present a utility-based model that evaluates ESO *costs* by accounting for the employee's hedging strategies and early exercises, along with the common ESO features of vesting and sudden job termination risk. Since the ESO cost crucially depends on the timing of exercise, we have to first examine the employee's optimal exercise policy and the associated utility maximization problem. To this end, we apply indifference pricing theory to derive and analyze the employee's optimal exercising strategy. This leads to the study of a joint *stochastic control* and *optimal stopping* problem. In a diffusion model, we describe the employee's optimal exercise policy in terms of an optimal exercise boundary obtained from solving the associated free boundary problem (see Section 2.1 and 3.1). This boundary then becomes an input for the firm's option pricing problem, in which a risk-neutral pricing model is used to compute the ESO cost (see Section 2.2).

By incorporating job termination risk, we obtain a nonlinear free boundary problem of reaction-diffusion type for the employee's investment problem. Reaction-diffusion equations also arise in utility problems in incomplete markets, for example, indifference pricing with interacting Itô and point processes (Becherer (2004)), and utility valuation in credit risk (Sircar and Zariphopoulou (2006)). Furthermore, we find that the higher the job termination risk, the earlier the employee exercises the ESO (see Figure 3.1(left)), leading to a significantly lower ESO cost (Figure 4.2).

In Section 5, we consider the case of ESOs with multiple exercise opportunities. The employee's

utility-maximization leads to a *system* of free boundary problems, and his optimal exercising strategy is characterized by a collection of optimal exercise boundaries (see Figure 5.1). We illustrate that a risk averse employee will gradually exercise fractions of their options through maturity, a pattern observed in empirical studies on ESOs (see Huddart and Lang (1996)). Several other utility-based models also observe partial exercises of American options in incomplete markets. For perpetual American ESOs with zero interest rate, Grasselli and Henderson (2008) provide an analytic formula for the employee’s exercise thresholds. For American ESOs with finite maturity, Grasselli (2005) and Rogers and Scheinkman (2007) numerically determine the employee’s optimal exercise policy, but in the absence of vesting and job termination risk which are incorporated in our model here. We also analyze the optimal exercise policy as well as the ESO costs for various grant sizes. In particular, when a vesting period is imposed, the cost of ESOs with multiple exercise rights and the cost with simultaneous exercise constraint are almost the same (see Figure 5.2).

As standard options have become more liquid over the past decade, they have emerged as alternative instruments for hedging derivatives. In Section 6, we consider hedging strategies that include both *dynamic* trading of a correlated asset and *static* positions in market-traded options. This combined strategy, which is referred to as *static-dynamic* hedging (see for example İlhan and Sircar (2005), and Leung and Sircar (2008)), allows us to study how market prices of traded options affect the employee’s optimal hedging and exercising strategy. In the case of static-dynamic hedging of a single ESO with vanilla put options, we find that the employee’s optimal ESO exercise time is significantly delayed (see Figure 6.1). Effectively, the static put position makes the employee’s less risk averse by offering downside protection. Moreover, we compute the optimal static hedge of this combined strategy from the Fenchel-Legendre transform of the indifference price as a function of the number of options used in the static hedge evaluated at their market prices. The incorporation of static hedges can drastically increase the ESO costs compared with case with only dynamic hedge (see Section 6.1), but the costs remain significantly lower than the Black-Scholes price.

In Section 7, we provide a formulation for static-dynamic hedging of ESOs with multiple exercises. With some simplifications, this complicated problem can be modeled in a similar way as the case with only dynamic hedge as discussed in Section 5. This allows us to derive a similar system of free boundary problems of reaction-diffusion type, and numerically solve using a method discussed in Leung and Sircar (2007).

## 2 The ESO Valuation Model

In this section, we present our valuation model for a single ESO, in which the employee partially hedges his ESO position by dynamically investing in the bank account and a correlated market index. We call this the *dynamic* hedge model to distinguish it from the *static-dynamic* hedge model to be presented in Section 6.

To start with, let us consider a riskless bank account that pays interest at constant rate  $r$ , and two risky assets, namely, the firm's stock and a market index. The employee can only trade the bank account and the market index, but not the firm's stock. The latter is modeled as a diffusion process that satisfies

$$dY_u = (\nu - q)Y_u du + \eta Y_u dW_u, \quad u \geq t,$$

with  $Y_t = y > 0$ . The coefficients  $\nu$ ,  $q$  and  $\eta$  are constant parameters, representing respectively the stock's expected return, dividend rate, and volatility. The market index is another lognormal process that is partially correlated with the firm's stock

$$dS_u = \mu S_u du + \sigma S_u dB_u, \quad u \geq t,$$

with  $S_t = S > 0$ . The two Brownian motions  $B$  and  $W$  are defined on a probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_u), \mathbb{P})$ , where  $\mathcal{F}_u$  is the augmented  $\sigma$ -algebra generated by  $\{W_s, B_s; 0 \leq s \leq u\}$ , and their instantaneous correlation is  $\rho \in (-1, 1)$ . The employee can use  $S$  to partially hedge away some of the risk in their portfolio, with some remaining idiosyncratic risk.

The employee stock option in this paper is an American call option on the firm's stock with maturity  $T$  (typically 10 years), with strike  $K$  and a vesting period  $t_v \leq T$  (typically 2 to 4 years). At the exercise time, the firm sells a new stock issue to the employee at the price  $K$ . Following the arguments in Hull and White (2004) and FASB statement 123R, we work under the assumption that the possible dilution effect is anticipated by the market and already reflected in the stock price immediately after the ESO grant.

Due to vesting, the employee cannot exercise the option before  $t_v$ . If the employee leaves the firm during the vesting period, then the option becomes worthless. If the employee's departure happens after vesting, then he must exercise the ESO if it is in-the-money. As vesting period lengthens to

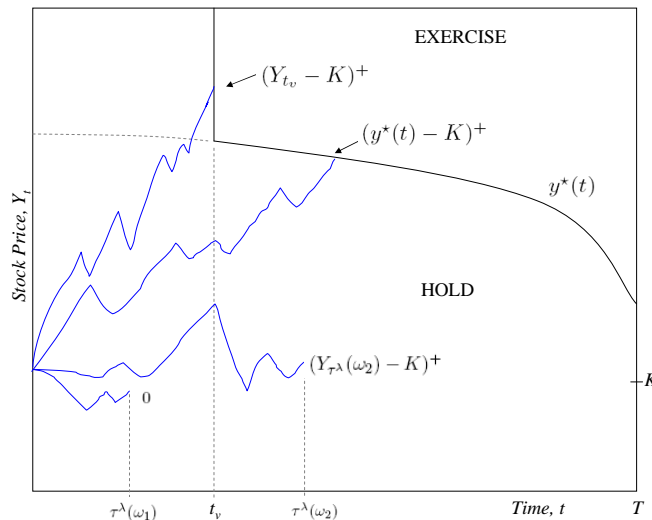


Figure 2.1: **ESO payoff structure.** The bottom path represents the scenario where the employee leaves the firm during the vesting period, resulting in forfeiture of the ESO. In the next path above, the employee is forced to exercise the ESO early due to job termination. The second from the top path hits the optimal exercise boundary  $y^*(t)$  after vesting, so the employee exercises the ESO there. On the top path, the employee exercises the ESO immediately at the end of vesting.

maturity, the ESO becomes a European call - the holder can exercise the ESO only at maturity.

The modeling of job termination is a delicate and important issue that has a crucial impact on ESO valuation, as we demonstrate in Figure 4.2. The fact that the horizon of the valuation problem is typically much shorter than the contractual term of the ESO has even been recognized in the FASB proposal, in which it recommends that the ESO maturity be shortened according to the job termination risk. On the one hand, it would be nice to develop and estimate a detailed model to account for the causes of job termination that separate voluntary and involuntary exits, and the classification of employees, for example, by age. In particular, external opportunities that tempt the employee to depart and exercise the ESO early might be considered. On the other hand, data is scarce and likely not well-segmented according to the identity of employee, or even the cause of job termination. Therefore, the literature has adopted reduced-form modeling that bypasses direct modeling of an individual employee's personal employment choices and potential inducement from external offers. Models that involve more complex information, including the fortellability of the employee's voluntary exit, are topics for future development as more comprehensive empirical data becomes available.

In our model, the employee's (voluntary or involuntary) employment termination time, denoted

by  $\tau^\lambda$ , is represented by an exponential random variable with parameter  $\lambda$  that is independent of the Brownian motions  $W$  and  $B$ . Leung and Sircar (2007) address how to adapt this formulation to more complex  $\tau^\lambda$ . The rate of job termination  $\lambda$  can be estimated from the firm's historical data. For instance, one can take the inverse of the average time to job termination. We illustrate the payoff structure of the ESO in Figure 2.1.

## 2.1 The Employee's Investment Problem

Since the employee cannot sell the ESO, or form a perfect hedge, it is important to consider his risk aversion. To this end, we represent his risk preference with the exponential utility function  $U(x) = -e^{-\gamma x}$ , with a positive constant absolute risk aversion  $\gamma$ .

To solve the employee's investment problem, it is sufficient to consider the case with zero vesting. When vesting increases from zero, it effectively lifts the employee's *pre-vesting* exercise boundary to infinity, but leaves his *post-vesting* exercise policy unaffected. Now suppose, at time  $t \in [0, T]$ , the employee is endowed with an ESO and some positive wealth. The employee's investment problem is to decide *when* to exercise the option. We define  $\mathcal{T}_{t,T}$  as the set of stopping times (with respect to the filtration  $(\mathcal{F}_u)$ ) taking values in  $[t, T]$ . Throughout the entire period  $[t, T]$ , the employee is assumed to trade dynamically in the bank account and the market index. A trading strategy  $\{\theta_u; t \leq u \leq T\}$  is the cash amount invested in the market index  $S$ , and it is deemed admissible if it is  $\mathcal{F}_u$ -progressively measurable and satisfies the integrability condition  $\mathbb{E} \left\{ \int_t^T \theta_u^2 du \right\} < \infty$ . The set of admissible strategies over the period  $[t, T]$  is denoted by  $\Theta_{t,T}$ . For  $u \geq t$ , the employee's trading wealth evolves according to

$$(2.1) \quad dX_u^\theta = [\theta_u(\mu - r) + rX_u] du + \theta_u \sigma dB_u, \quad X_t = x.$$

Upon the exercise of the option, either voluntarily or forced due to job termination, the employee will add the contract proceeds to his portfolio, and continue to optimally invest in the bank account and market index up to the maturity date  $T$ . Therefore, from the exercise time till the expiration date, the employee, who no longer holds an ESO, faces the classical Merton problem of optimal investment. According to Merton (1969), if an investor has  $x$  dollars at time  $t \leq T$  and invests dynamically in the bank account and the market index until time  $T$ , then his maximal expected

utility is given by

$$\begin{aligned}
(2.2) \quad M(t, x) &= \sup_{\Theta_{t,T}} \mathbb{E} \left\{ -e^{-\gamma X_T} \mid X_t = x \right\} \\
&= -e^{-\gamma x e^{r(T-t)}} e^{-\frac{(\mu-r)^2}{2\sigma^2}(T-t)}.
\end{aligned}$$

We can think of this as the employee's utility of having wealth  $\$x$  at time  $t \leq T$ . The ESO holder's investment problem is formulated as a stochastic utility maximization with optimal stopping. We shall use the following shorthands for conditional expectations:

$$\mathbb{E}_{t,y} \{ \cdot \} = \mathbb{E} \{ \cdot \mid Y_t = y \}, \quad \mathbb{E}_{t,x,y} \{ \cdot \} = \mathbb{E} \{ \cdot \mid X_t = x, Y_t = y \}.$$

The employee's value function at time  $t \in [0, T]$ , given that he has not departed the firm and that his wealth  $X_t = x$  and firm's stock price  $Y_t = y$ , is

$$\begin{aligned}
(2.3) \quad V(t, x, y) &= \sup_{\tau \in \mathcal{T}_{t,T}} \sup_{\Theta_{t,\tau}} \mathbb{E}_{t,x,y} \left\{ M(\hat{\tau}, X_{\hat{\tau}} + (Y_{\hat{\tau}} - K)^+) \right\} \\
&= \sup_{\tau \in \mathcal{T}_{t,T}} \sup_{\Theta_{t,\tau}} \mathbb{E}_{t,x,y} \left\{ -e^{-\gamma(X_{\hat{\tau}} + (Y_{\hat{\tau}} - K)^+)e^{r(T-\hat{\tau})}} e^{-\frac{(\mu-r)^2}{2\sigma^2}(T-\hat{\tau})} \right\},
\end{aligned}$$

where  $\hat{\tau} = \tau \wedge \tau^\lambda$ . Observe that we are explicitly optimizing the expected utility over all stopping times, and over all trading strategies  $\theta$  before  $\tau$ . The *post-exercise* trading is implicitly optimized by the solution to the Merton problem  $M$ . Both of the expectations in (2.2) and (2.3) are taken under the historical measure,  $\mathbb{P}$ . By standard arguments from the theory of optimal stopping, the employee's optimal exercise time is given by

$$(2.4) \quad \tau^* := \inf \{ t \leq u \leq T : V(u, X_u, Y_u) = M(u, X_u + (Y_u - K)^+) \}.$$

## 2.2 ESO Cost to the Firm

The employee's optimal exercise time and the corresponding exercise boundary can be obtained by solving a free boundary problem to be discussed in the next section. Meanwhile, let us explain how to use the employee's exercise boundary to determine the ESO cost to the firm. In accordance with

the FASB rules<sup>2</sup>, we model the firm’s stock with the following dynamics

$$dY_u = (r - q)Y_u du + \eta Y_u dW_u^{\mathbb{Q}}, \quad u \geq t; \quad Y_t = y,$$

where  $W^{\mathbb{Q}}$  is a Brownian motion under the risk-neutral measure  $Q$ , which is also independent of the job termination time  $\tau^\lambda$ . As in Carr and Linetsky (2000), job termination rate is assumed to be identical under both measures  $\mathbb{P}$  and  $\mathbb{Q}$ ; that is, the job termination risk is unpriced. By no-arbitrage arguments, the firm’s granting cost is given by the no-arbitrage price of a barrier-type call option subject to early exercise due to job termination. The barrier is the employee’s optimal exercise boundary. It is possible that the employee will leave the firm before the vesting period ends, or job termination arrives before the stock reaches the optimal boundary. In the first case, the ESO is forfeited. In the latter case, the employee is forced to exercise the option immediately. We must consider both cases in order to accurately determine the ESO value to the firm.

We first consider the cost of an *vested* ESO. Suppose the vesting period is  $t_v$  years. At time  $t \geq t_v$ , given that the stock price  $Y_t$  is  $y$  and the ESO is still alive, the cost of the ESO is given by

$$(2.5) \quad \begin{aligned} C(t, y) &= \mathbb{E}_{t,y}^{\mathbb{Q}} \left\{ e^{-r(\tau^* \wedge \tau^\lambda - t)} (Y_{\tau^* \wedge \tau^\lambda} - K)^+ \right\} \\ &= \mathbb{E}_{t,y}^{\mathbb{Q}} \left\{ e^{-(r+\lambda)(\tau^* - t)} (Y_{\tau^*} - K)^+ + \int_t^{\tau^*} e^{-(r+\lambda)(u-t)} \lambda (Y_u - K)^+ du \right\}. \end{aligned}$$

Next, we consider the *unvested* ESO. Let  $\tilde{C}(t, y)$  be the cost of the unvested ESO at time  $t \leq t_v$  given that it is still alive and the stock price  $Y_t = y$ . It is given by

$$(2.6) \quad \tilde{C}(t, y) = \mathbb{E}_{t,y}^{\mathbb{Q}} \left\{ e^{-r(t_v - t)} C(t_v, Y_{t_v}) \mathbf{1}_{\{\tau^\lambda > t_v\}} \right\}.$$

To efficiently compute the vested and unvested ESO costs, we solve the corresponding partial differential equation problems for  $C(t, y)$  and  $\tilde{C}(t, y)$  using a finite-difference method. We refer the reader to Leung and Sircar (2007) for a detailed formulation and a numerical solution method. In Section 4, we will analyze the main contributors to the ESO costs.

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<sup>2</sup>In paragraph A13 of FASB 123R, it specifically requires the use of “techniques that are used to establish trade prices for derivative instruments,” and approves the use of risk-neutral models. Even if a firm does not hedge its ESOs, it should calculate and report the cost generated from such models.

### 3 The Employee's Exercise Policy

The employee's optimal exercise boundary is not known *ex ante*; it has to be inferred from the solution to the free boundary problem associated with the value function  $V$ . We proceed to determine the employee's optimal exercise boundary, and provide a characterization for it. Afterward, we will investigate how various parameters influence the employee's exercising strategy.

#### 3.1 The Free Boundary Problem of Reaction-Diffusion Type

Before deriving the free boundary problem for the employee's investment problem, let us introduce the following differential operators

$$\mathcal{L} = \frac{\eta^2 y^2}{2} \frac{\partial^2}{\partial y^2} + \rho \theta \sigma \eta y \frac{\partial^2}{\partial x \partial y} + \frac{\theta^2 \sigma^2}{2} \frac{\partial^2}{\partial x^2} + (\nu - q)y \frac{\partial}{\partial y} + [\theta(\mu - r) + rx] \frac{\partial}{\partial x},$$

which is the infinitesimal generator of  $(X, Y)$ , and

$$\tilde{\mathcal{L}} = \frac{\eta^2 y^2}{2} \frac{\partial^2}{\partial y^2} + (\nu - q - \rho \frac{\mu - r}{\sigma} \eta)y \frac{\partial}{\partial y},$$

which is the infinitesimal generator of  $Y$  under the minimal entropy martingale measure (see Leung and Sircar (2007) for details). Also, we define the utility rewarded for immediate exercise

$$\Lambda(t, x, y) = M(t, x + (y - K)^+) = -e^{-\gamma(x + (y - K)^+)e^{r(T-t)}} e^{-\frac{(\mu - r)^2}{2\sigma^2}(T-t)}.$$

By dynamic programming principle, the value function  $V$  is conjectured to solve the following Hamilton-Jacobi-Bellman variational inequality

$$\lambda(\Lambda - V) + V_t + \sup_{\theta} \mathcal{L}V \leq 0,$$

$$(3.1) \quad V \geq \Lambda,$$

$$\left( \lambda(\Lambda - V) + V_t + \sup_{\theta} \mathcal{L}V \right) \cdot (\Lambda - V) = 0,$$

for  $(t, x, y) \in [0, T] \times \mathbb{R} \times (0, +\infty)$ . The boundary conditions are

$$V(T, x, y) = -e^{-\gamma(x+(y-K)^+)}, \quad V(t, x, 0) = -e^{-\gamma x e^{r(T-t)}} e^{-\frac{(\mu-r)^2}{2\sigma^2}(T-t)}.$$

Due to the exponential utility function (see also Oberman and Zariphopoulou (2003)), this problem can be simplified by a separation of variables and power transformation

$$(3.2) \quad V(t, x, y) = M(t, x) \cdot H(t, y)^{\frac{1}{(1-\rho^2)}}.$$

The free boundary problem for  $H$  is of *reaction-diffusion* type.

$$(3.3) \quad H_t + \tilde{\mathcal{L}} H - (1 - \rho^2)\lambda H + (1 - \rho^2)\lambda b(t, y)H^{-\hat{\rho}} \geq 0,$$

$$H(t, y) \leq \kappa(t, y),$$

$$\left( H_t + \tilde{\mathcal{L}} H - (1 - \rho^2)\lambda H + (1 - \rho^2)\lambda b(t, y)H^{-\hat{\rho}} \right) \cdot \left( \kappa(t, y) - H(t, y) \right) = 0,$$

for  $(t, y) \in [0, T] \times (0, +\infty)$ , where

$$\hat{\rho} = \frac{\rho^2}{1 - \rho^2}, \quad b(t, y) = e^{-\gamma(y-K)^+ e^{r(T-t)}}, \quad \text{and} \quad \kappa(t, y) = e^{-\gamma(1-\rho^2)(y-K)^+ e^{r(T-t)}}.$$

The boundary conditions are

$$(3.4) \quad H(T, y) = e^{-\gamma(1-\rho^2)(y-K)^+}, \quad H(t, 0) = 1.$$

This problem for  $H$  implies that the employee's optimal exercise time is independent of  $X$  and  $S$ . Therefore, we define the employee's optimal exercise boundary as the function  $y^* : [0, T] \mapsto \mathbb{R}_+$ , where  $y^*(t)$  is the critical stock price at time  $t$ . That is,

$$y^*(t) = \inf\{y \geq 0 : H(t, y) = \kappa(t, y)\}.$$

In practice, we numerically solve this free boundary problem to obtain the employee's exercise boundary  $y^*$ . Then, the employee's optimal exercise time is the first time that the firm's stock

reaches  $y^*$ . That is,

$$(3.5) \quad \tau^* = \inf\{t \leq u \leq T : Y_u = y^*(u)\}.$$

### 3.2 Characterization of the Employee's Exercise Boundary

The function  $H$ , defined in (3.2), turns out to be related to the employee's indifference price for the ESO, which will allow us to characterize the employee's optimal exercise time. We are primarily interested in the *cost* of an ESO to the firm, not the employee's indifference price. Nevertheless, the indifference price is a useful concept in analyzing the employee's exercise behavior.

**Definition 3.1** *The ESO holder's indifference price of an ESO (without vesting) is defined as the function  $p \equiv p(t, x, y)$  such that  $M(t, x) = V(t, x - p, y)$ .*

Due to the exponential utility function, the indifference price is a function of only  $t$  and  $y$ . By Definition (3.1) and the transformation (3.2), one can deduce the following fact.

**Proposition 3.2** *The employee's indifference price for the ESO, denoted by  $p$ , satisfies*

$$(3.6) \quad p(t, y) = -\frac{1}{\gamma(1 - \rho^2)e^{r(T-t)}} \log H(t, y).$$

Consequently, it solves the following variational inequality

$$(3.7) \quad \min\left\{-p_t - \tilde{\mathcal{L}}p + rp + \frac{1}{2}\gamma(1 - \rho^2)\eta^2 y^2 e^{r(T-t)} p_y^2 + \frac{\lambda}{\gamma} \left(b(t, y)e^{\gamma p e^{r(T-t)}} - 1\right), p - (y - K)^+\right\} = 0,$$

with  $p(T, y) = (y - K)^+$ , and  $p(t, 0) = 0$ .

Applying Definition 3.1 to (2.4), we characterize the optimal exercise time  $\tau^*$  in terms of  $p$ :

$$(3.8) \quad \tau^* = \inf\{t \leq u \leq T : p(u, Y_u) = (Y_u - K)^+\}.$$

This provides a nice interpretation for the employee's optimal exercising strategy: he will exercise the ESO as soon as the payoff from immediate exercise equals his indifference price. Moreover, the connection between the two expressions for  $\tau^*$  in (3.5) and (3.8) provides a channel to understand the employee's exercise boundary  $y^*$  through analyzing properties of the indifference price  $p$ .

### 3.3 Effects of Parameters on the Employee’s Exercise Policy

In this section, we summarize several important results on the behavior of the employee’s optimal exercise boundary. The analysis relies on studying the variational inequality for  $p$  in (3.7) and its connection with  $\tau^*$  in (3.5) and (3.8). Detailed proofs and analysis as well as a numerical solution method for the exercise boundary can be found in Leung and Sircar (2007).

First, let us first study the effect of job termination risk. Figure 3.1 (left) shows that higher job termination risk induces the employee to adopt a more conservative exercising strategy (a lower exercise boundary). In other words, the employee fears the *possibility* of sudden job termination (and a possible loss of ESO value), so he exercises the ESO earlier to lock in profit.

Empirical studies on ESOs (Huddart and Lang (1996)) show that most ESO holders exercise well before the options expire. Our model offers some explanation to this phenomenon. First, Figure 3.1 illustrates that early exercise is optimal for a risk-averse employee. Also, job termination risk induces the employee to lower his exercise boundary, leading to even earlier exercise. Lastly, when job termination actually happens prior to maturity, then the employee is forced to give up or exercise the ESO. All these contribute to the early exercise phenomenon.

In our numerical example depicted in Figure 3.1 (right), the employee’s exercise boundary shifts downward as risk aversion increases. This agrees with intuition because a higher risk aversion implies a greater tendency to lock in sure profit now, rather than waiting for a possibly higher return in the future. Hence, a more risk-averse holder would exercise the option at a lower critical price.

In the ESO valuation models proposed by Hull and White (2004), and Cvitanic et al. (2004), the employee’s exercise boundary is exogenously specified and does not change with the dividend rate, drift, and volatility of the firm’s stock. Empirical studies have shown that these parameters influence the employee’s exercise behavior. For example, Bettis et al. (2005) point out that ESOs are exercised earlier in firms with higher dividend yields. This is reasonable because a higher dividend rate entices the employee to own the firm’s stock share and receive the dividend. In our model, we observe that the exercise boundary shifts downward as dividend rate increases, or as the firm’s stock’s expected return decreases. Moreover, as shown in Leung and Sircar (2007), the exercise boundary has non-monotonic interaction with the volatility parameter  $\eta$ , correlation  $\rho$ , and Sharpe ratio of the market index.

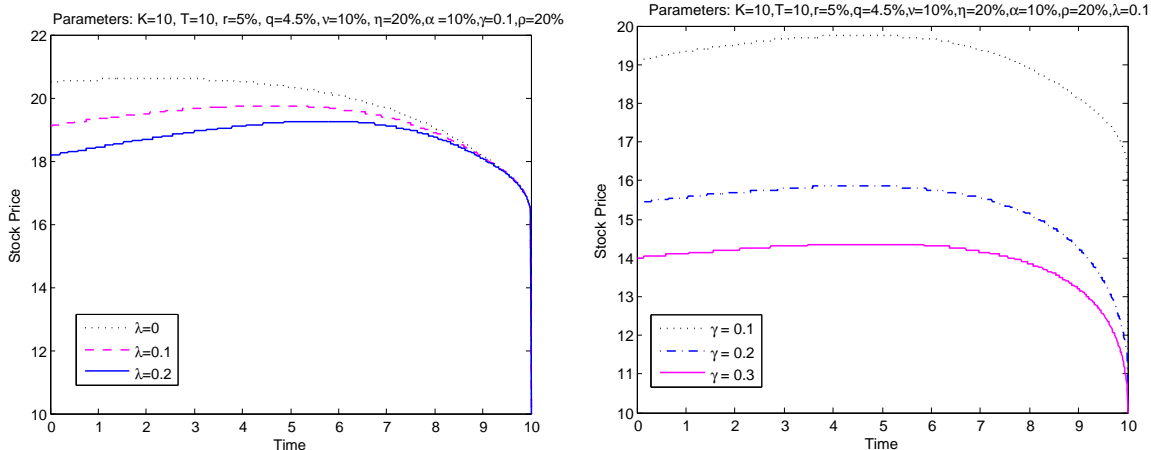


Figure 3.1: *Effects of job termination risk and risk aversion: (Left) Higher job termination risk lowers the exercise boundary. (Right) As risk aversion  $\gamma$  increases, the employee's exercise boundary moves downward.*

## 4 Analysis of the ESO Cost

With reference to the definitions (2.5) and (2.6), the ESO cost can be computed using partial differential equation methods. Here, we study the effects of risk aversion, vesting, and job termination risk on the cost of an ESO, and compare our results with other models. We refer to the reader to Leung and Sircar (2007) for detailed discussion and proofs.

### 4.1 Effects of Vesting, Risk Aversion, and Job Termination Risk

We first analyze the effects of vesting and risk aversion in the absence of job termination risk. Recall that the risk-averse employee's optimal exercise boundary, which maximizes his expected utility from exercising the ESO, in general does not maximize the expected discounted return of the ESO, i.e. the ESO cost. From the firm's perspective, the employee's risk-averse attitude means that the ESO costs *less* than the no-arbitrage price of the corresponding American call. If the risk aversion is small, then the ESO cost would be between the values of an American call and a European call on the firm's stock with the same strike and maturity. However, as the employee becomes more risk averse, his utility-maximizing boundary shifts downward, getting further away from the price-maximizing boundary. Consequently, the cost of the ESO decreases as risk aversion increases. If the ESO holder is sufficiently risk-averse, the cost of the ESO to the firm could be even lower than a European call on the firm's stock with the same strike and maturity.

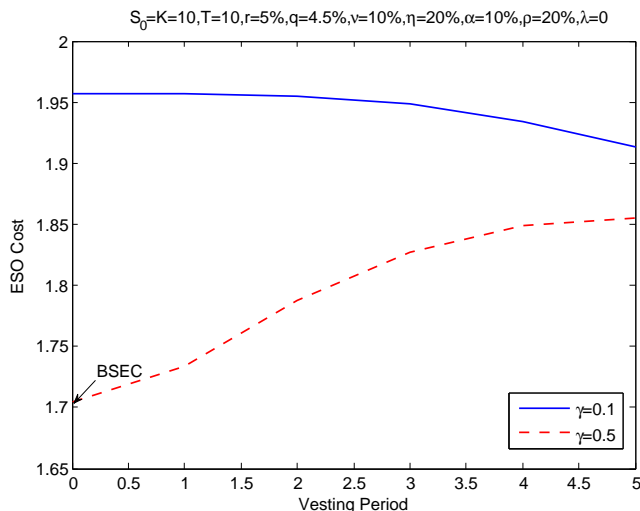


Figure 4.1: **Effect of vesting:** The marker “BSEC” represents the no-arbitrage price of a European call with the same strike and maturity as the ESO. In the absence of job termination risk, the cost of an ESO held by a very risk-averse employee increases (from close to BSEC) with respect to vesting. In the low risk aversion case, the cost decreases with vesting but stays above BSEC.

If the firm imposes vesting on the ESO, then any exercise before the end of the vesting period is prevented. Effectively, the pre-vesting part of the employee’s utility-maximizing boundary is lifted to infinity. Since vesting imposes discipline on the employee which restrains the employee’s risk-averse behavior, it could increase the expected discounted payoff, implying a higher cost to the firm (see Figure 4.1). We can prove this for the case of no dividend and no job termination risk.

**Proposition 4.1** *If  $\lambda = q = 0$ , then the ESO cost is non-decreasing with respect to the length of the vesting period. Moreover, this cost is dominated by the Black-Scholes price of the European call option written on firm’s stock with the same strike and maturity.*

The consideration of the employee’s risk aversion gives us an important insight to the cost structure of ESOs to the firm – vesting may involve additional cost. While the firm may be able to maintain the incentive effect of the ESOs and impose discipline on ESO exercises, they may also have to pay for these benefits. This is not reflected in risk-neutral ESO valuation models. If the employee were to hedge perfectly and thus were risk-neutral, then the ESO cost would certainly decrease with vesting.

In Figure 4.2, we observe that higher job termination risk reduces the ESO cost. This is because the employee adopts a more conservative exercising strategy in the presence of job termination risk,

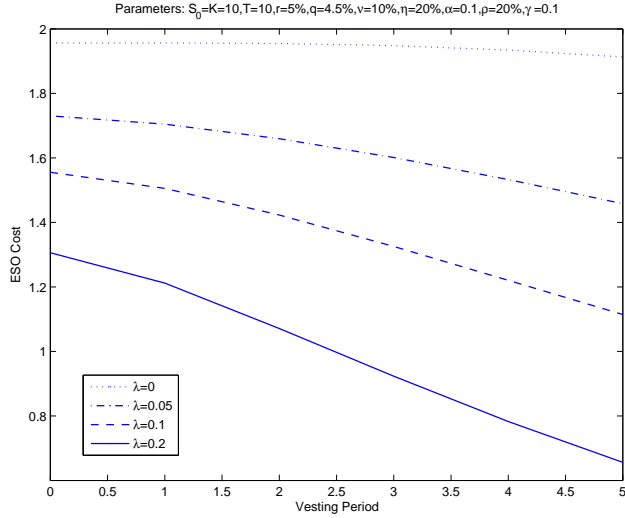


Figure 4.2: **Effect of job termination risk:** The ESO cost decreases significantly as the job termination risk rises. Furthermore, the cost decreases as vesting period lengthens due to the increasing likelihood of forfeiture.

while possible departure may result in either forfeiture of the option or early suboptimal exercise. All these factors drive the ESO cost lower. Similarly, as the vesting period lengthens, the employee becomes more likely to depart before vesting ends, leading to a much lower ESO cost (Figure 4.2).

## 4.2 Comparison with Other Models

We compare our model with the ones proposed by Henderson (2007) and Grasselli (2005). Henderson (2007) assumes an infinite horizon with zero interest rate and no job termination risk. The resulting exercise threshold (a flat boundary) tends to be very high, and only exists under some parameter restrictions. Taking into account positive interest rate and finite maturity, but not job termination risk, Grasselli (2005) obtains a lower cost. Our model incorporates the risk of job termination and vesting, which further reduce the ESO cost. The following table shows the different ESO costs under the parameter values given in Figure 4.3. The first entry is the Black-Scholes price of a ten-year European option, which in this case of no dividend is equal to the American price. The next two entries are from Henderson (2007) and Grasselli (2005) models specialized to just one option. The last two entries add job termination risk and then vesting.

Black-Scholes	Henderson	Grasselli	+ $\lambda = 0.1$	+3-yr vesting
4.878	4.510	3.412	2.597	2.491

We observe that risk-aversion lowers the cost by about 8% in the perpetual approximation, or by about 30% when we retain finite maturity, but then job termination risk reduces the cost by a further 17% of the Black-Scholes value, and vesting by yet another 2% in this example.

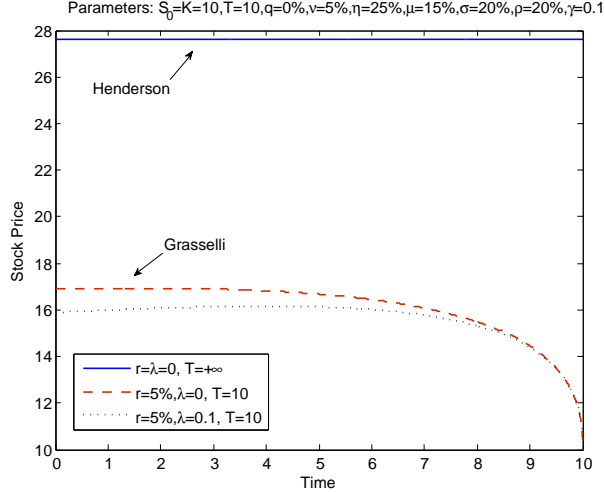


Figure 4.3: *Comparing exercise boundaries:* The model by Henderson (2007) gives a high flat exercise boundary. Grasselli (2005) corresponds to the middle boundary. The bottom boundary from our model accounts for the presence of job termination risk. The parameter values are chosen so that the exercise boundary by Henderson (2007) is finite.

## 5 Valuation Model With Multiple Exercises

We extend the model to the case in which the employee is granted multiple ESOs which may be exercised separately. In particular, we are interested in characterizing the employee’s optimal exercise strategy. Suppose that, at time  $t \in [0, T]$ , the employee is granted  $n$  American-styled ESOs with the same strike and maturity. Denote by  $\tau_i$  the exercise time when  $i$  options remain unexercised. We require that  $\tau_i \in \mathcal{T}_{t, T}$ , and clearly, we have  $\tau_n \leq \dots \leq \tau_1$ . If the employee exercises multiple options at the same time, then some exercise times may coincide.

Throughout the period  $[t, T]$ , the employee dynamically invests his wealth in the bank account and the market index  $S$ , so his trading wealth follows (2.1). At every discretionary exercise time,  $\tau_i$ , the employee invests the option payoff into his trading portfolio. However, at the job termination time  $\tau^\lambda$ , he must exercise *all remaining options*. After exiting from the firm, the employee is assumed to invest the contract proceeds into his trading portfolio and continue trading till time  $T$ . Given

that the employee has not departed the firm and holds  $i \geq 2$  unexercised options at time  $t$ , his value function is given by

$$(5.1) \quad V^{(i)}(t, x, y) = \sup_{t \leq \tau_i \leq T} \sup_{\Theta_{t, \tau_i}} \mathbb{E}_{t, x, y} \left\{ V^{(i-1)}(\tau_i, X_{\tau_i} + (Y_{\tau_i} - K)^+, Y_{\tau_i}) \cdot \mathbf{1}_{\{\tau_i < \tau^\lambda\}} + M(\tau^\lambda, X_{\tau^\lambda} + i(Y_{\tau^\lambda} - K)^+) \cdot \mathbf{1}_{\{\tau_i \geq \tau^\lambda\}} \right\},$$

with  $V^{(1)} = V$  in (2.3) and  $V^{(0)} = M$  in (2.2). This leads to a chain of stochastic control problems with optimal multiple stopping. To determine the optimal exercise boundaries, we solve the system of variational inequalities for the value functions  $\{V^{(i)}\}$ , which can be simplified via a transformation like that in (3.1) to a free boundary problem similar to that for  $H$  in (3.3). Then, the costs can be computed using no-arbitrage arguments with these exercise boundaries as inputs. A detailed discussion can be found in Leung and Sircar (2007). Here, we summarize the important effects of multiple exercises on both the optimal exercise policy and the ESO costs of various grant sizes.

## 5.1 The Impact of Multiple Exercises

We first study the effect of multiple exercises on the employee's exercise policy. In the traditional no-arbitrage pricing theory for American options, the option holder always exercises all the options at the same time. In our model, the risk-averse employee exercises his ESOs at different critical price levels (see Figure 5.1). This is because the employee's premium for an additional option diminishes with respect to the number of options he already owns. As a result, the employee tends to exercise the first option very early, and the last one much later. Similar exercise behaviors can be found in Grasselli (2005), Rogers and Scheinkman (2007) and Grasselli and Henderson (2008).

In order to study the impact of multiple exercises, we compare it with the case with simultaneous exercise constraint. This constraint allows the employee to choose only one exercise time for all his ESOs. This is equivalent to the single issue case with the ESO payoff multiplied by the number of options. As Figure 5.1 illustrates, his boundary lies somewhere in the middle.

Finally, we examine the effect of multiple issues on the firm's granting cost. As the number of ESOs increases, the employee tends to adopt a more conservative exercising strategy for every additional option, which in turns results in a diminishing marginal cost. This is depicted in Figure 5.2. When there is no vesting, the cost of ESOs with simultaneous exercise constraint dominates

that with multiple exercise rights (see Figure 5.2 (left)). This is because the simultaneous exercise constraint prevents the employee from exercising too early, leading to a higher expected discounted payoff.

However, when a two-year vesting period is imposed, the costs are almost the same (see Figure 5.2 (right)). In the case of multiple exercises, the majority of the employee’s exercise boundaries are very low, so it is very likely that the firm’s stock will be above most of them, leading to exercises at the end of the vesting period. Similarly, in the constrained case, the low boundary implies that the employee will probably exercise all his ESOs at the end of the vesting period. This result is consistent with the well-known early exercise phenomenon in ESO empirical studies. As a result, in the presence of vesting, the right of multiple exercises has negligible influence on the firm’s granting cost.

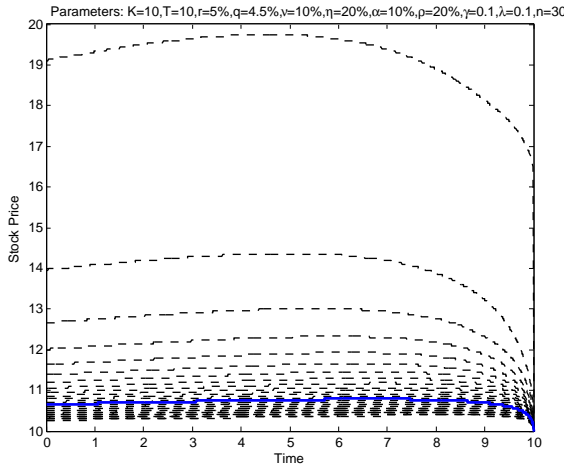


Figure 5.1: **Multiple exercise boundaries:** The dashed curves represent the exercise boundaries for an employee with 30 ESOs with multiple exercise rights. The bottom one is the optimal exercise boundary for the first option. Higher curves are the exercise boundaries for the subsequent options. The boundaries for the first few options are very low and concentrated, suggesting that the employee tends to exercise a fraction of options very early. When the employee is granted 30 ESOs with simultaneous exercise constraint, his exercise boundary (solid line) lies somewhere in the middle.

## 6 Static-Dynamic Hedges for ESOs and Impact on Valuation

In this section, we consider the ESO holder’s hedging strategy that combine dynamic trading of correlated assets and static positions in market-traded put options written on the firm’s stock.

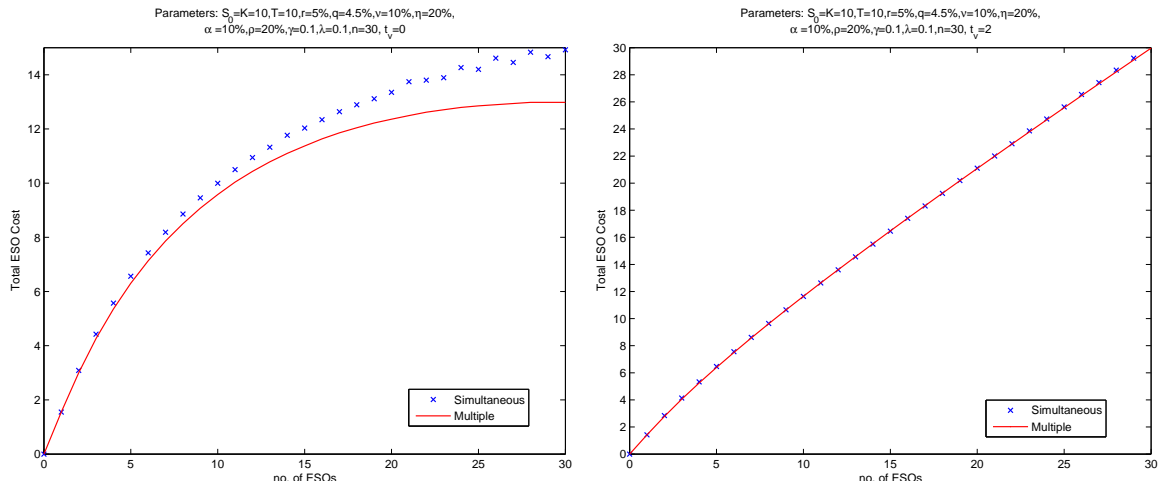


Figure 5.2: **Effect of multiple exercises:** (Left) When there is no vesting, the cost of ESOs with simultaneous exercise constraint dominates that with multiple exercise rights. (Right) However, when vesting is imposed, the difference in costs almost disappears.

Initially, the employee is granted an ESO, and uses some of his initial wealth to buy and hold some standard put options. In addition to holding the put options, the employee also chooses an optimal dynamic trading strategy in a correlated tradable asset and the bank account, funded by his remaining capital. For a detailed discussion on static-dynamic hedging of American options in a general (semimartingale) incomplete financial market, we refer the reader to Leung and Sircar (2008).

As an example, let us suppose that the ESO holder purchases from the market  $\alpha$  units of American puts with payoff  $\alpha(K' - Y_\tau)^+$  at an exercise time  $\tau$  at the cost of  $\$ \pi'$  each. For simplicity, here we assume no job termination risk and that all puts are exercised simultaneously. In order to decide which option(s) (ESO or puts) to exercise first, the employee needs to compare the value functions for holding the ESO only and for holding the American puts only. The former is given by the value function  $V(t, x, y)$  in (2.3), and the value function for holding only  $\alpha$  American puts is given by

$$V^D(t, x, y; \alpha) := \sup_{\tau \in \mathcal{T}_{t,T}} \sup_{\Theta_{t,\tau}} \mathbb{E}_{t,x,y} \{ M(\tau, X_\tau + \alpha(K' - Y_\tau)^+) \}.$$

Also, let  $\{p^D(t, y; \alpha), \tau_t^D(\alpha), y^D\}$  be the corresponding indifference price, optimal exercise time, and optimal exercise boundary. Due to its similarity with  $V$ , the problem for  $V^D$  and all its associated quantities can be solved using the same techniques for  $V$  as discussed in Section 3.

Then, by Proposition 4.2 of Leung and Sircar (2008), the employee's value function for statically hedging the ESOs with  $\alpha$  American puts is given by  $\hat{V}(t, x - \alpha\pi', y; \alpha)$  where

$$\begin{aligned}\hat{V}(t, x, y; \alpha) &= \sup_{\tau \in \mathcal{T}_{t,T}} \sup_{\Theta_{t,\tau}} \mathbb{E}_{t,x,y} \{ \max(V(\tau, X_\tau + \alpha(K' - Y_\tau)^+; A), V^D(\tau, X_\tau + (Y_\tau - K)^+; \alpha)) \} \\ &= \sup_{\tau \in \mathcal{T}_{t,T}} \sup_{\Theta_{t,\tau}} \mathbb{E}_{t,x,y} \{ M(\tau, X_\tau + R(\tau, Y_\tau; \alpha)) \} = V(t, x, y; R^\alpha),\end{aligned}$$

with  $R(t, y; \alpha) := \max\{(y - K)^+ + p^D(t, y; \alpha), \alpha(K' - y)^+ + p(t, y)\}$ . Hence, we have simplified the problem for  $\hat{V}$  to the form of  $V$  (defined in 2.3), with  $\lambda = 0$  and payoff  $R^\alpha(t, y; \alpha)$  instead of  $(y - K)^+$ . It also admits a separation of variables:

$$\hat{V}(t, x - \alpha\pi', y; \alpha) = M(t, x - \alpha\pi') \cdot \hat{H}(t, y; \alpha)^{\frac{1}{(1-\rho^2)}},$$

and  $\hat{H}$  satisfies a linear free boundary similar to  $H$  (with  $\lambda = 0$  and different boundary conditions). We numerically solve for each fixed  $\alpha \geq 0$  for the indifference price  $\hat{p}(t, y; \alpha)$  and optimal exercise boundaries (see Figure 6.1). Then, the optimal quantity of American puts to purchase  $\alpha^*$  is found from the Fenchel-Legendre transform of the indifference price  $\hat{p}(t, y; \alpha)$  as a function of  $\alpha$ , evaluated at the market price  $\pi'$  (see also Leung and Sircar (2008)). That is,

$$(6.1) \quad \alpha^* = \arg \max_{0 \leq \alpha < \infty} \hat{p}(t, y; \alpha) - \alpha\pi'.$$

The optimal time to exercise the first option(s) (ESO or puts) is given by

$$\hat{\tau}_t^*(\alpha) = \inf \{ t \leq u \leq T : \hat{p}(u, Y_u; \alpha) = R(u, Y_u; \alpha) \} = \min\{\tau_t^{AD}(\alpha), \tau_t^{DA}(\alpha)\},$$

where

$$\begin{aligned}\tau_t^{AD}(\alpha) &:= \inf \{ t \leq u \leq T : \hat{p}(u, Y_u; \alpha) = (Y_u - K)^+ + p^D(u, Y_u; \alpha) \}, \\ \tau_t^{DA}(\alpha) &:= \inf \{ t \leq u \leq T : \hat{p}(u, Y_u; \alpha) = \alpha(K' - Y_u)^+ + p(u, Y_u) \},\end{aligned}$$

represent, respectively, the optimal times to exercise the ESO *first*, and the American puts *first*. As illustrated in Figure 6.1, they are characterized as the first time that the firm's stock hits the respective exercise boundaries,  $y^{AD}, y^{DA} : [0, T] \mapsto \mathbb{R}_+$ . In the first case (Figure 6.1 (left)), the stock reaches the boundary  $y^{AD}$  first, i.e.  $\hat{\tau}_t^* = \tau_t^{AD} \leq \tau_t^{DA}$ , and the employee will first exercise the ESO

at the boundary  $y^{AD}$ . After this ESO exercise, the employee will exercise the remaining American puts when the stock reaches the exercise boundary  $y^D$ . In the other scenario (Figure 6.1 (right)), the stock hits the boundary  $y^{DA}$  before hitting  $y^{AD}$ ; i.e.  $\hat{\tau}_t^* = \tau_t^{DA} \leq \tau_t^{AD}$ .

The positions in the ESO and American puts exhibit interactive effects on the risk-averse employee's optimal exercise times. Leung and Sircar (2008) show that  $\tau_t^{AD} \geq \tau_t^*$ , which implies that the boundary  $y^{AD}$  (with static-dynamic hedge) always *dominates*  $y^*$  (with only dynamic hedge). As shown in Figure 6.1, the employee's optimal exercise boundary for the ESO is lifted upward when American puts are used. This means that static-dynamic hedges with American puts always *delay* ESO exercise compared to the exercise time with dynamic hedge only. On the other hand, the long ESO position also induces the employee to delay his American put exercise, i.e.  $\tau_t^{DA} \geq \tau_t^D$ .

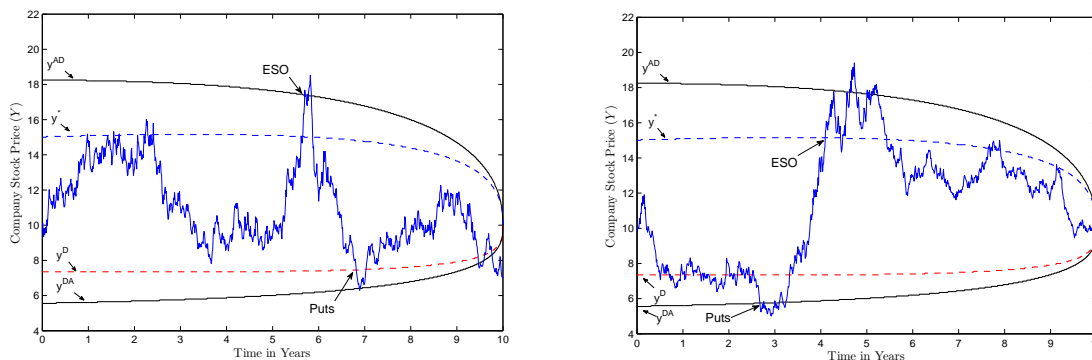


Figure 6.1: *The order of exercises is determined by whether the stock price hits the upper or lower solid boundary first. (Left) If the stock price first touches the upper solid boundary ( $y^{AD}$ ), then the employee exercises the ESO here, and then the puts later when it crosses the lower dashed boundary ( $y^D$ ). (Right) If the stock price touches the lower solid boundary first ( $y^{DA}$ ), then the employee exercises all the puts here, and then the ESO later when it crosses the upper dashed boundary ( $y^*$ ). Moreover,  $y^{AD}$  (with static-dynamic hedge) always dominates  $y^*$  (with only dynamic hedge), implying that the static-dynamic hedge delays ESO exercise. The parameters are  $K = K' = 10$ ,  $T = 10$ ,  $r = 5\%$ ,  $q = 0\%$ ,  $\nu = 8\%$ ,  $\eta = 30\%$ ,  $(\mu - r)/\sigma = 20\%$ ,  $\rho = 30\%$ ,  $\gamma = 0.3$ , Black-Scholes put price  $\pi' = 1.322$ , with  $\alpha^* = 1.32$ .*

## 6.1 The Impact of Static-Dynamic Hedges on ESO costs

Once we obtain the optimal exercise boundaries, we can use them to compute the ESO cost using standard no-arbitrage pricing argument. Next, we present a numerical example to illustrate the impacts of various hedging strategies on the ESO cost to the firm. The following table compares the ESO costs for the cases with and without vesting under the parameter values given in Figure

6.1. The entries in the last column are the Black-Scholes price of a ten-year at-the-money American option on the firm's stock. They are the same under different vesting provisions because the optimal time to exercise is at expiry (due to zero dividend rate). The entries in the second column are the costs when the employee hedges only dynamically as in Section 2 with zero job termination risk. The next column is from the case of static-dynamic hedges with American puts shown in Figure 6.1.

	Dynamic Hedge	Static-dynamic Hedge with American Puts	Black-Scholes (Perfect Hedge)
$t_v = 0$	3.0330	3.6931 ( $\alpha^* = 1.32$ )	5.2567
$t_v = 2$	3.4376	3.8396 ( $\alpha^* = 1.25$ )	5.2567
$t_v = 4$	3.9932	4.1941 ( $\alpha^* = 1.07$ )	5.2567

The Black-Scholes model gives the highest cost across different vesting periods because it assumes the ESO holder can perfectly hedge. When the employee can only dynamically hedge with the market index, the ESO cost is only 58% of the Black-Scholes value in the case with no vesting period. We notice that the ESO cost increases as the employee adopts a more effective hedging strategy. For instance, in the first row, static hedges American puts increase the costs by 22% compared with the cost from dynamic hedge, but they remain significantly lower than the Black-Scholes price. The situation is similar when vesting is imposed, but vesting drives the ESO cost closer to the Black-Scholes value by preventing the employee from exercising early. In the limit of  $t_v = T$ , the ESO becomes a European option and it can only be exercised at expiry. The cost under various hedging strategies will coincide with the Black-Scholes value.

## 7 Static-Dynamic Hedges for ESOs with Multiple Exercises

In Section 6, we have formulated a basic framework for static-dynamic hedging of an ESO, in which the static hedging instruments are all exercised at once. The problem of static-dynamic hedging of *multiple* ESOs using other *multiple* exercising American claims can also be formulated similarly using the principle of dynamic programming. To write down the value function, one has to first consider the maximal expected utilities from all possible orders of exercises. As the number of multiple exercising claims increases, the value function becomes very tedious, though straightforward, to write down. Moreover, the resulting optimal exercising strategies will also be too complex to describe. As an approximation, one can limit the number of exercise opportunities to a small finite number, or just adopt the case with simultaneous exercise or static-dynamic hedges with European claims.

In the final part of this paper, we provide a formulation for static-dynamic hedging of *multiple* ESOs with *multiple* European puts, along with job termination risk. In this case, the ESO holder purchases  $\beta \geq 0$  European puts with strike  $K'$  and maturity  $T$ . By Theorem 3 in Musiela and Zariphopoulou (2004), the indifference price for holding the European puts can be written as

$$h(t, y; \beta) = -\frac{1}{\gamma(1-\rho^2)e^{r(T-t)}} \log \tilde{\mathbb{E}} \left\{ e^{-\gamma(1-\rho^2)\beta(K'-Y_T)^+} | Y_t = y \right\},$$

where the expectation is taken under the minimal entropy martingale measure. It can be obtained by solving a linear partial differential equation (see Leung and Sircar (2008)). Then, following the setup in Section 5, we can define recursively the value function for an employee holding  $i \geq 2$  ESOs and  $\beta$  European puts, while subject to job termination risk, as

$$\begin{aligned} \tilde{V}^{(i)}(t, x, y; \beta) = \sup_{t \leq \tau_i \leq T} \sup_{\Theta_{i, \tau_i}} \mathbb{E}_{t, x, y} \left\{ \tilde{V}^{(i-1)}(\tau_i, X_{\tau_i} + (Y_{\tau_i} - K)^+ + h(\tau_i, Y_{\tau_i}; \beta), Y_{\tau_i}; \beta) \cdot \mathbf{1}_{\{\tau_i < \tau^\lambda\}} \right. \\ \left. + M \left( \tau^\lambda, X_{\tau^\lambda} + i(Y_{\tau^\lambda} - K)^+ + h(\tau^\lambda, Y_{\tau^\lambda}; \beta) \right) \cdot \mathbf{1}_{\{\tau_i \geq \tau^\lambda\}} \right\}. \end{aligned}$$

This again leads to a system of free boundary problems of reaction-diffusion type. We observe the similarity between the value functions  $\{\tilde{V}^{(i)}\}$  and  $\{V^{(i)}\}$  (in (5.1)). In fact, their underlying free boundary problems differ only in the boundary conditions, so the problem for  $\{\tilde{V}^{(i)}\}$  can be numerically solved by adopting the numerical method for  $\{V^{(i)}\}$  discussed in Leung and Sircar (2007). As in (6.1), the optimal static hedge is found from the Fenchel-Legendre transform of the indifference price for  $\tilde{V}^{(i)}$ . We have seen that job termination risk induces early exercises while static hedges tend to delay exercises. Hence, it would be of theoretical interest and practical importance to examine the interaction between job termination risk and optimal static hedges, and their combined effect on ESO costs.

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